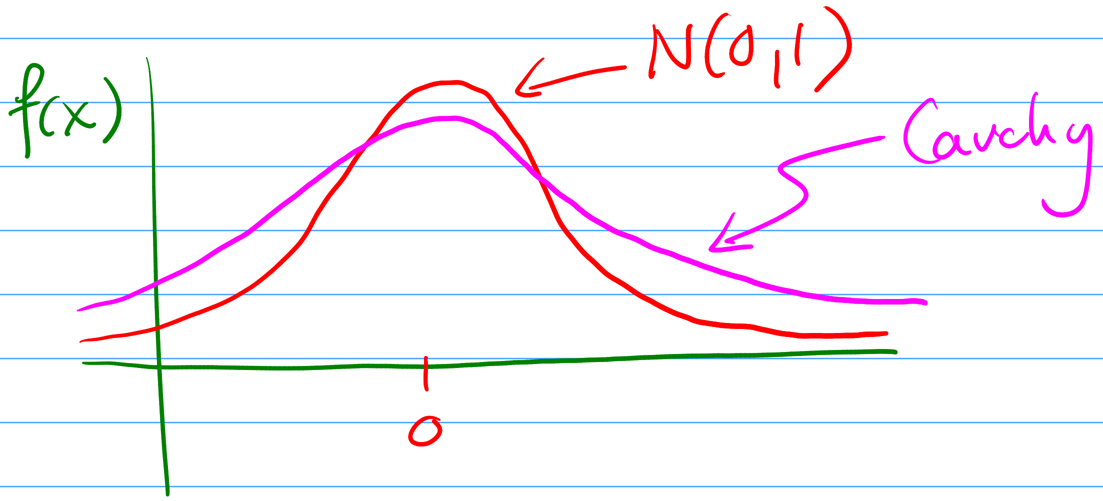
Lecture 10:

Ex. Cauchy Distribution

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2} \quad \text{for } x \in \mathbb{R}^2$$



$$E[X] = \int_{\mathbb{R}} x f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\pi} \frac{1}{1+x^2}\, dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx$$

$$\sim \frac{x}{x^2} = \frac{1}{x}$$

$$\sim \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x}\, dx = \infty$$

$$\sum_{i=1}^{\infty} \frac{1}{i} = \infty$$

Say: the mean doesn't exist.

Theorem: Properties of Expectation

① Expectation is linear:

$$E[aX+b] = aE[X]+b.$$

pf (cts)

$$E[aX+b] = \int (ax+b)f(x)\,dx$$

$$= \int (axf(x) + bf(x))\,dx$$

$$= \underbrace{a\int xf(x)\,dx}_{E[X]} + \underbrace{b\int f(x)\,dx}_{1}$$

$$= aE[X]+b.$$

② If $\underbrace{X \geq 0}$ then $E[X] \geq 0$.

→ support is non-neg

pf (cts)

$$E[X] = \int_0^\infty \overbrace{\underbrace{x}_{\geq 0}\,\underbrace{f(x)}_{\geq 0}}^{\geq 0}\,dx \geq 0.$$

③ If g_1 and g_2 are functions

ⓘ $E[g_1(X) + g_2(X)] = E[g_1(X)] + E[g_2(X)]$

ⓘⓘ $g_1(x) \leq g_2(x)$ then

$$E[g_1(X)] \leq E[g_2(X)]$$

④ If $a \leq X \leq b$ then

$$a \; E[X] \leq b.$$

$\mu = E[X]$ = loc. of dist

$\sigma^2 = Var(X)$ = spread of dist. around mean

Defn: Variance

Let $\mu = E[X]$

then the variance is

$$Var(X) = E[\underbrace{(X-\mu)^2}_{\text{expected sq. dist. of } X \text{ from its mean}}]$$

$$= E[(X - E[X])^2]$$

units are on squared scale

Defn: Standard Deviation

$$SD(X) = \sqrt{Var(X)}.$$

Ex. $X \sim Exp(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0$$

$$\mu = E[X] = 1/\lambda, \quad E[X^2] = \frac{2}{\lambda^2}$$

$$Var(X) = E[(X-\mu)^2]$$

$$= \int_0^\infty (x-\mu)^2 f(x)\, dx$$

$$= \int_0^\infty (x-\mu)^2 \lambda e^{-\lambda x} dx$$

correct: but difficult

$$\begin{aligned} Var(X) &= E[(X-\mu)^2] \\ &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 \\ &= \frac{2}{\lambda^2} - 2\left(\frac{1}{\lambda}\right)\left(\frac{1}{\lambda}\right) + \left(\frac{1}{\lambda}\right)^2 \end{aligned}$$

$$\boxed{Var(X) = 1/\lambda^2}$$

Theorem: Short-Cut Formula for Variance

$$\boxed{Var(X) = E[X^2] - E[X]^2.}$$

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pf. $Var(X) = E[(X-\mu)^2]$ $\qquad \boxed{\mu = E[X]}$

$$\begin{aligned} &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 \\ &= E[X^2] - 2E[X]^2 + E[X]^2 \\ &= E[X^2] - E[X]^2. \end{aligned}$$

Ex. $X \sim Exp(\lambda)$

$E[X] = 1/\lambda$ and $E[X^2] = 2/\lambda^2$

So $Var(X) = E[X^2] - E[X]^2$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= 1/\lambda^2.$$

Ex. $X \sim Bin(n, p)$

$E[X] = np$

$E[X^2] = np(np + 1 - p)$

$Var(X) = E[X^2] - E[X^2]$

$$= np(np + 1 - p) - (np)^2$$

$$= np(\cancel{np} + 1 - p - \cancel{np})$$

$$= np(1-p).$$

Theorem:

$$\boxed{\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X).}$$

① multply by a ⇝ Var mult. by a^2

② add b ⇝ Var ignores

pf. $\mathrm{Var}(aX+b)$

$= E[(aX+b)^2] - E[aX+b]^2$

$= E[a^2X^2 + 2abX + b^2] - (aE[X]+b)^2$

$= a^2E[X^2] + \cancel{2abE[X]} + \cancel{b^2} - [a^2E[X]^2 \cancel{-2abE[X]} + \cancel{b^2}]$

$= a^2(E[X^2] - E[X]^2)$

$= a^2\,\mathrm{Var}(X).$