Lecture 11 : Moments

Defn: Moment

If r is a pos. integer then the r^{th} moment of X is

$$\mu_r = E[X^r].$$

Ex. $\mu_1 = E[X] = \mu$

$\mu_2 = E[X^2]$

$\mu_3 = E[X^3]$

$\vdots$

Defn: Moment Generating Function (MGF)

If X is a RV then its MGF is a function

$$M : \mathbb{R} \to \mathbb{R}$$

defined for $t \in \mathbb{R}$ as

$$M(t) = E[e^{tX}].$$

$E[X]$ $\quad E[g(X)]$ $\quad g(x) = e^{tx}$

discrete:

$$M(t) = E[e^{tX}] = \sum_x e^{tx} f(x)$$

cts:

$$M(t) = E[e^{tX}] = \boxed{\int_{\mathbb{R}} e^{tx} f(x)\, dx}$$

X - a r.v.

x - a number

$$E[g(X)] = \int_{\mathbb{R}} g(x) f(x)\, dx$$

Ex. $X \sim Exp(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

$$M(t) = E[e^{tX}]$$

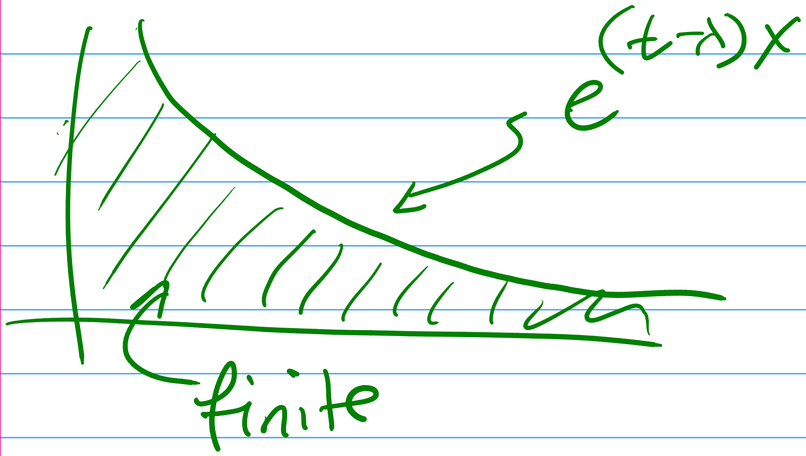
$$= \int_{\mathbb{R}} e^{tx} f(x)\, dx$$

$$= \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$\boxed{e^a e^b = e^{a+b}}$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x} dx$$

If $\underline{t-\lambda < 0} \iff t < \lambda$



If $\underline{t-\lambda \geq 0 \iff t \geq \lambda}$

Consider $t < \lambda$

then

$$M(t) = \lambda \int_0^\infty e^{(t-\lambda)x} \, dx$$

$$= \lambda \left[\frac{e^{(t-\lambda)x}}{t-\lambda} \right]_0^\infty$$

$$= \lambda \left[\frac{0-1}{t-\lambda} \right] = \boxed{\frac{\lambda}{\lambda - t} = M(t) \text{ for } t<\lambda}$$

Consider:

$$\left.\frac{dM}{dt}\right|_{t=0} = \left.\frac{\lambda}{(\lambda-t)^2}\right|_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$= E[X]$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = \left.\frac{2\lambda}{(\lambda-t)^3}\right|_{t=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

$$= E[X^2]$$

Theorem:

$$\left.\frac{d^r M}{dt^r}\right|_{t=0} = M^{(r)}(0) = E[X^r] = \mu_r .$$

pf. recall:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{tX} = 1 + Xt + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots$$

$$M(t) = E[e^{tX}]$$

$$= 1 + tE[X] + \frac{t^2E[X^2]}{2!} + \frac{t^3E[X^3]}{3!} + \cdots$$

$$\left.\frac{dM}{dt}\right|_{t=0} = 0 + \underbrace{E[X]}_{\text{only thing left.}} + \frac{2\cancel{t}E[X^2]}{2!} + \frac{3\cancel{t^2}E[X^3]}{3!} + \cdots$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = 0 + 0 + E[X^2] + \frac{3\cdot 2\cdot \cancel{t}^{0} E[X^3]}{3!} + \cdots$$

$$= E[X^2]$$

<u>Ex.</u> $X \sim Bin(n, p)$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 0, 1, 2, \ldots, n$.

$$E[X] = np = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$E[X^2] = np + n(n-1)p^2$$

$$= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}$$

Binomial Theorem:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$M(t) = E[e^{tX}]$$

$$= \sum_{x} e^{tx} f(x)$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^{x} (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (\underbrace{pe^{t}}_{a})^{x} (\underbrace{1-p}_{b})^{n-x}$$

$$= (a+b)^{n}$$

$$\boxed{M(t) = (pe^{t} + 1 - p)^{n}}$$

$$E[X] = \frac{dM}{dt}\Big|_{t=0} = n(pe^{t} + 1 - p)^{n-1} \cdot pe^{t}\Big|_{t=0}$$

$$= n(pe^{0} + 1 - p)^{n-1} pe^{0}$$

$$= n(p + 1 - p)^{n-1} p$$

$$= np$$

Can similarly show that

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = np + n(n-p)p^2 .$$

$$= E[X^2]$$

Theorem: If $Y = aX + b$ then

$$M_Y(t) = e^{tb} M_X(at) .$$

Pf.

$$M_Y(t) = E[e^{tY}]$$

$$= E[e^{t(aX+b)}]$$

$$= E[e^{atX} e^{tb}]$$

$$= e^{tb} E[e^{(at)X}]$$

$$= e^{tb} M_X(at).$$

Comon Distributions

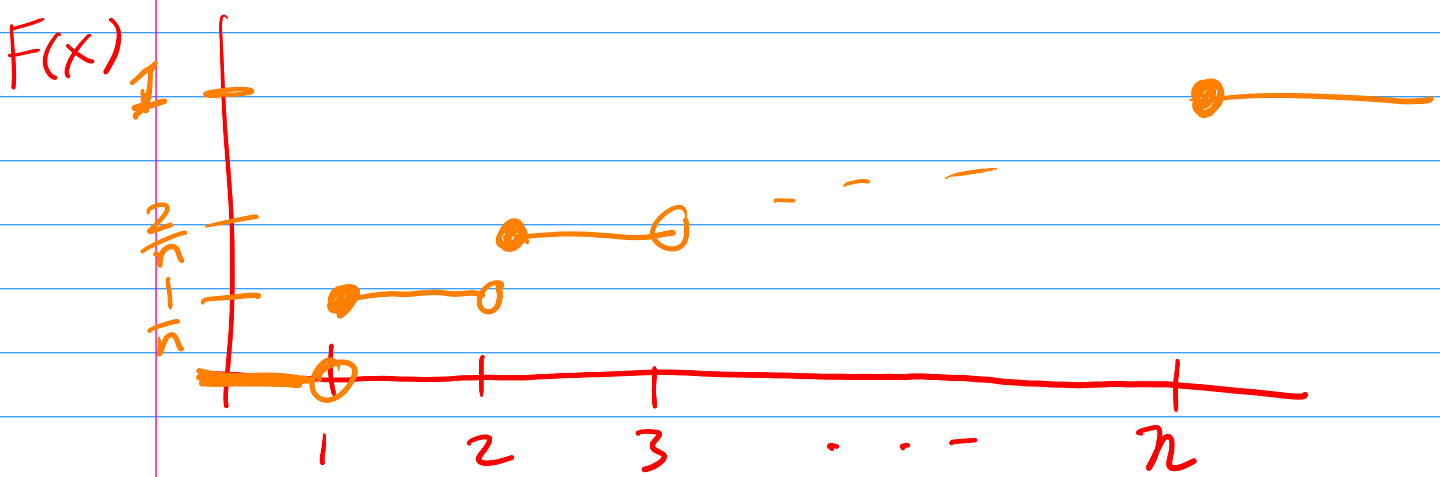
Discrete Uniform

$X \sim U(\{1, \dots, n\})$

PMF: $\boxed{f(x) = \frac{1}{n} \quad \text{for } x = 1, \dots, n}$

CDF:

$$F(x) = \begin{cases} 0, & x < 1 \\ \lfloor x \rfloor / n, & 1 \le x \le n \\ 1, & x > n \end{cases}$$



$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$E[X] = \sum_{x} x f(x)$$

$$= \sum_{x=1}^{n} x \frac{1}{n} = \frac{1}{n} \sum_{x=1}^{n} x = \frac{1}{n} \frac{n(n+1)}{2}$$

$$= \frac{n+1}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$E[X^2] = \sum_{x} x^2 f(x)$$

$$= \sum_{x=1}^{n} x^2 \frac{1}{n} = \frac{1}{n} \sum_{x=1}^{n} x^2 = \frac{(n+1)(2n+1)}{6}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= \left(\frac{(n+1)(2n+1)}{6}\right) - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{n^2-1}{12}$$

MGF:

$$M(t) = E[e^{tx}]$$

$$= \sum_x e^{tx} f(x)$$

$$= \sum_{x=1}^{n} e^{tx} \frac{1}{n}$$

$$= \frac{1}{n} \sum_{x=1}^{n} (e^t)^x$$

$$= \frac{1}{n} \sum_{x=0}^{n-1} (e^t)^{x+1}$$

$$= \frac{e^t}{n} \sum_{x=0}^{n-1} (\overbrace{e^t}^{r})^x = \frac{e^t}{n} \left(\frac{1-(e^t)^n}{1-e^t} \right)$$

$$e^t \neq 1$$

Geometric Sum:

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}, \quad r \neq 1$$

$$\boxed{M(t) = \frac{e^t - e^{(n+1)t}}{n(1-e^t)}, \quad t \neq 0}$$