Lecture 11: Moments

Defn: Moment

If r is a positive integer we define the r^{th} moment as

$$\mu_r = E[X^r].$$

Defn: Moment Generating Function (MGF)

If X is a RV then its MGF is a function

$$M : \mathbb{R} \to \mathbb{R}$$

defined for $t \in \mathbb{R}$ as

$$M(t) = E[e^{tX}].$$

Discrete: $M(t) = \sum_x e^{tx} f(x)$

cts: $\boxed{M(t) = \int_{\mathbb{R}} e^{tx} f(x)\,dx}$

Ex. $X \sim Exp(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0$$

$$M(t) = E[e^{tX}]$$

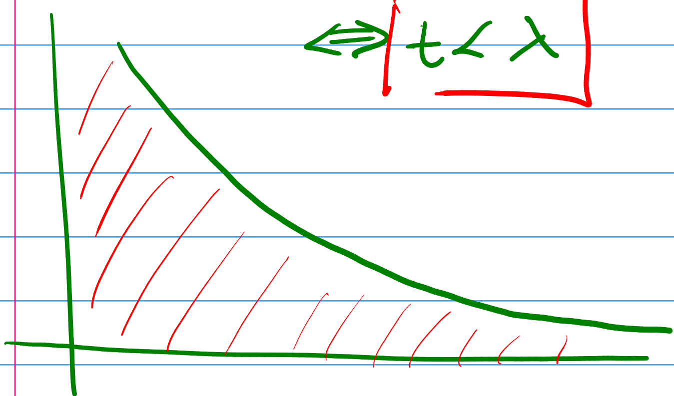
$$= \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x}\,dx$$

$t - \lambda < 0$

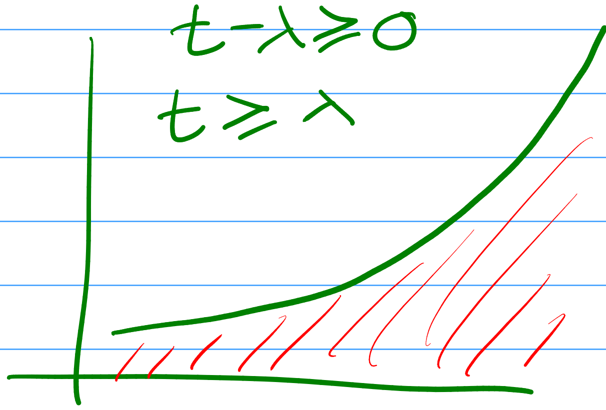
$\Leftrightarrow \boxed{t < \lambda}$



finite

$t - \lambda \geq 0$

$t \geq \lambda$



infinite

If $t < \lambda$ then

$$M(t) = \lambda \int_0^\infty e^{(t-\lambda)x}\,dx$$

$$= \lambda \left[\frac{e^{(t-\lambda)x}}{t-\lambda} \right]_0^{\infty}$$

$$= \frac{\lambda}{t-\lambda} [0 - 1] = \boxed{\frac{\lambda}{\lambda - t} \quad \text{for } t < \lambda}$$

Consider

$$\left.\frac{dM}{dt}\right|_{t=0} = \left.\frac{\lambda}{(\lambda-t)^2}\right|_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$= E[X]$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = \left.\frac{2\lambda}{(\lambda-t)^3}\right|_{t=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

$$= E[X^2]$$

Theorem:

$$\left.\frac{d^rM}{dt^r}\right|_{t=0} = E[X^r].$$

pf. Recall: $e^{x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2} + \frac{t^3X^3}{3!} + \cdots$$

$$M(t) = E[e^{tX}]$$

$$= 1 + t\,E[X] + \frac{t^2}{2}E[X^2] + \frac{t^3}{3!}E[X^3] + \cdots$$

$$\left.\frac{dM}{dt}\right|_{t=0} = 0 + (1)E[X] + \underbrace{\cancel{\frac{2t}{2}}^{\,0}E[X^2] + \frac{3t^2}{3!}E[X^3] + \cdots}$$

$$= E[X]$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = 0 + 0 + (1)\underbrace{E[X^2]} + \cancel{\frac{3\cdot 2\cdot t}{3!}}^{\,0}E[X^3] + \cdots$$

$$= E[X^2]$$

Ex. $X \sim Bin(n, p)$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 0, 1, 2, \ldots, n$

Claims: $E[X] = np$

$E[X^2] = np + n(n-1)p^2$

$$E[X] = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = \cdots = np$$

Binomial Theorem:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$M(t) = E[e^{tX}]$$

$$= \sum_x e^{tx} f(x)$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} \underbrace{(pe^t)}_{a}{}^{x} \underbrace{(1-p)}_{b}{}^{n-x}$$

$$= (a+b)^n$$

$$\boxed{M(t) = (pe^t + 1 - p)^n}$$

$$E[X] = \left.\frac{dM}{dt}\right|_{t=0} = \left. n(pe^t + 1 - p)^{n-1} pe^t \right|_{t=0}$$

$$= n(pe^0 + 1 - p)^{n-1} pe^0$$

$$= n(p + 1 - p)^{n-1} p$$

$$= np$$

Theorem: If $Y = aX + b$

$$M_Y(t) = e^{tb} M_X(at).$$

pf.

$$M_Y(t) = E[e^{tY}]$$

$$= E[e^{t(aX+b)}]$$

$$= E[e^{atX} e^{tb}]$$

$$= e^{tb} E[e^{(at)X}]$$

$$= e^{tb} M_X(at).$$

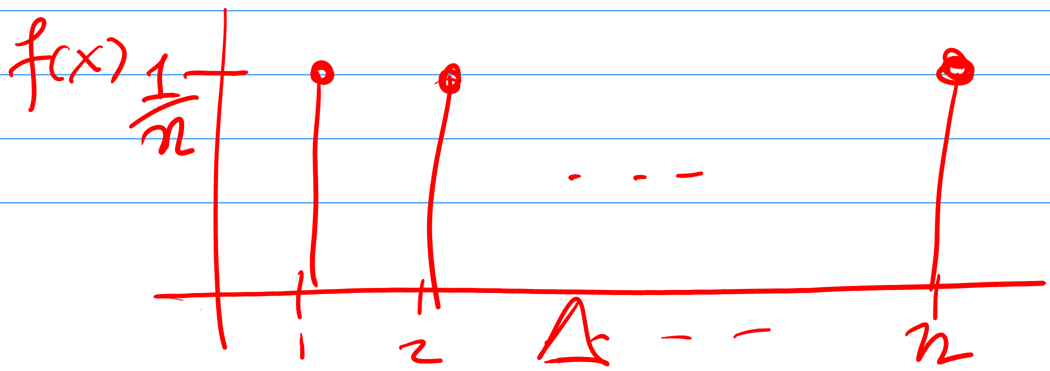
$$\boxed{e^a e^b = e^{a+b}}$$

Common Distributions

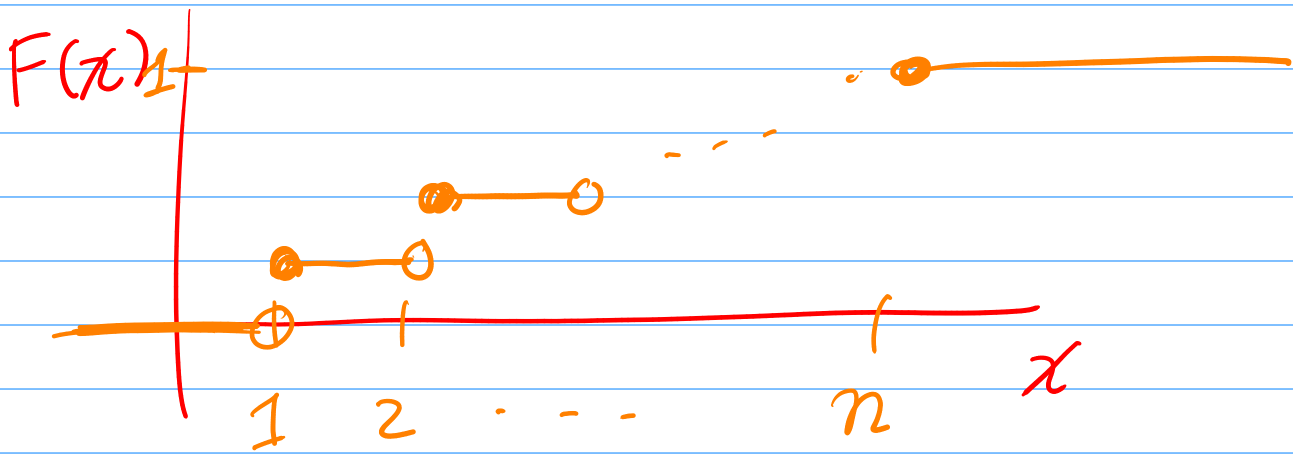
Discrete Uniform

$X \sim U(\{1, \dots, n\})$

$$f(x) = \frac{1}{n} \quad \text{for } x = 1, 2, 3, \dots, n$$



$$F(x) = \begin{cases} 0, & x < 1 \\ \lfloor x \rfloor / n, & 1 \le x \le n \\ 1, & x > n \end{cases}$$



$$E[X] = \sum_x x f(x)$$

$$= \sum_{x=1}^{n} x \frac{1}{n} = \frac{1}{n} \sum_{x=1}^{n} x = \frac{1}{n} \overbrace{\frac{(n+1)n}{2}}^{\binom{n+1}{2}}$$

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}} = \frac{n+1}{2}$$

$$E[X^2] = \sum_x x^2 f(x)$$

$$= \sum_{x=1}^{n} x^2 \frac{1}{n}$$

$$= \frac{1}{n}\sum_{x=1}^{n} x^2 = \frac{1}{\cancel{n}} \frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= \left(\frac{(n+1)(2n+1)}{6}\right) - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{n^2-1}{12}$$

MGF:

$$M(t) = E[e^{tX}]$$

$$= \sum_x e^{tx} f(x)$$

$$= \sum_{x=1}^{n} e^{tx} \frac{1}{n}$$

$$= \frac{1}{n} \sum_{x=1}^{n} (e^t)^x$$

Geometric Sum:

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}, \quad r \neq 1$$

$$= \frac{1}{n} \sum_{x=0}^{n-1} (e^t)^{x+1} \qquad e^t \neq 1$$

$$= \frac{e^t}{n} \underbrace{\sum_{x=0}^{n-1} (e^t)^x}_{r=e^t} = \frac{e^t}{n}\left(\frac{1-r^n}{1-r}\right)$$

$$= \frac{e^t}{n}\left(\frac{1-(e^t)^n}{1-e^t}\right)$$

$$\boxed{M(t) = \frac{e^t - e^{(n+1)t}}{n(1-e^t)} \quad \text{for } t \neq 0}$$