Lecture 12 : Common Dists

Generalize : $\boxed{Y \sim U(\{a, \dots, b\})}$

$n = b - a + 1$

$\boxed{X \sim U(\{1, \dots, n\})}$

then $Y \stackrel{d}{=} X + a - 1$

linear transf.

PMF: $f(y) = \frac{1}{n} = \frac{1}{b-a+1}$

for $y = a, a+1, \dots, b$

Expected Value:

$$E[Y] = E[X + a - 1]$$

$$= E[X] + a - 1$$

$$= \frac{n+1}{2} + a - 1$$

$$= \frac{(b-a+1)+1}{2} + a - 1 = \frac{a+b}{2}$$

Variance

$$Var(Y) = Var(X + a - 1)$$
$$= Var(X)$$
$$= \frac{n^2 - 1}{12}$$
$$= \frac{(b-a+1)^2 - 1}{12}$$

$$M_{aX+b}(t) = e^{tb} M_X(at)$$

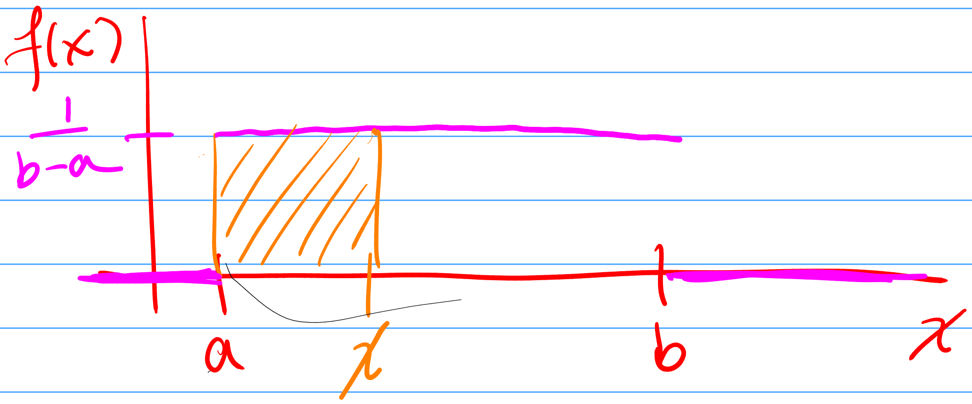
MGF:

$$M_Y(t) = M_{X+a-1}(t)$$
$$= e^{t(a-1)} M_X(t)$$
$$= e^{t(a-1)} \frac{e^t - e^{t(n+1)}}{n(1-e^t)}$$
$$= e^{t(a-1)} \frac{e^t - e^{t(b-a+2)}}{(b-a+1)(1-e^t)}$$

Continuous Uniform

$X \sim U(a, b)$

PDF: $f(x) = \dfrac{1}{b-a}$ for $a < x < b$

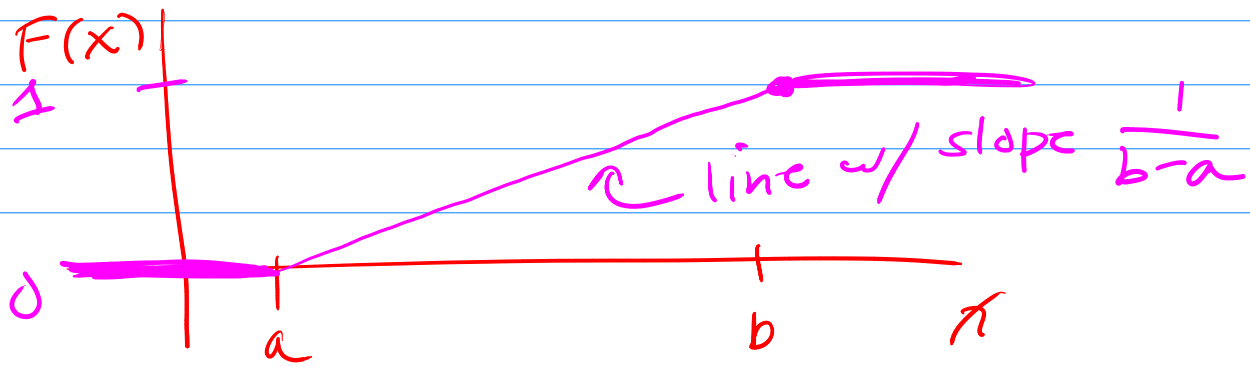


CDF:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{a}^{x} \frac{1}{b-a}\,dt$$

$a < x < b$

$$= \frac{t}{b-a}\Bigg|_{a}^{x} = \frac{x-a}{b-a}$$



Expectation

$$E[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{x^2}{2(b-a)}\Bigg|_a^b$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$$= \frac{a+b}{2}$$

$$E[X^2] = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{x^3}{3}\frac{1}{b-a}\Bigg|_a^b$$

$$= \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{\cancel{(b-a)}(a^2+ab+b^2)}{3\cancel{(b-a)}}$$

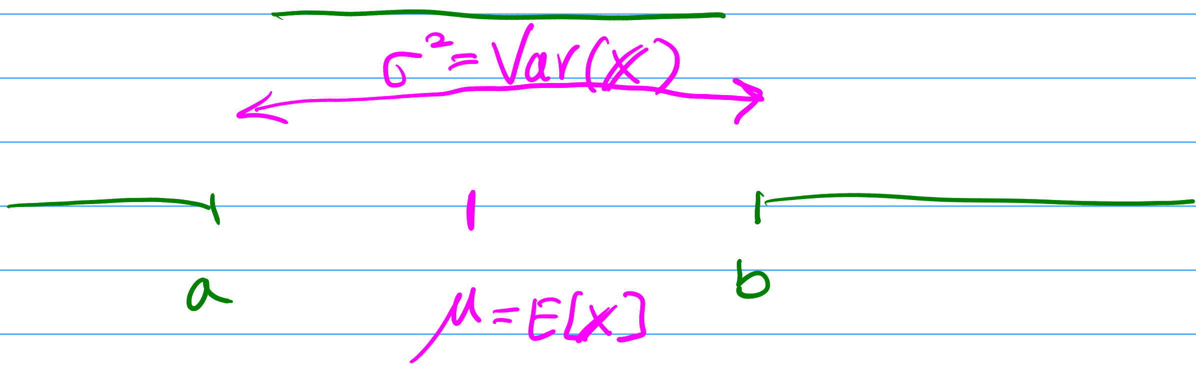
$$E[X^2] = \frac{a^2+ab+b^2}{3}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= \left(\frac{a^2+ab+b^2}{3}\right) - \left(\frac{a+b}{2}\right)^2$$

$$= \cdots$$

$$= \frac{(b-a)^2}{12}$$



MGF:

$$\boxed{M(t) = E[e^{tX}]} = \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\,dx$$

$$M(0) = E[e^0] = 1.$$

$t \neq 0$

$$= \frac{1}{b-a} \frac{1}{t} e^{tx} \Big|_a^b$$

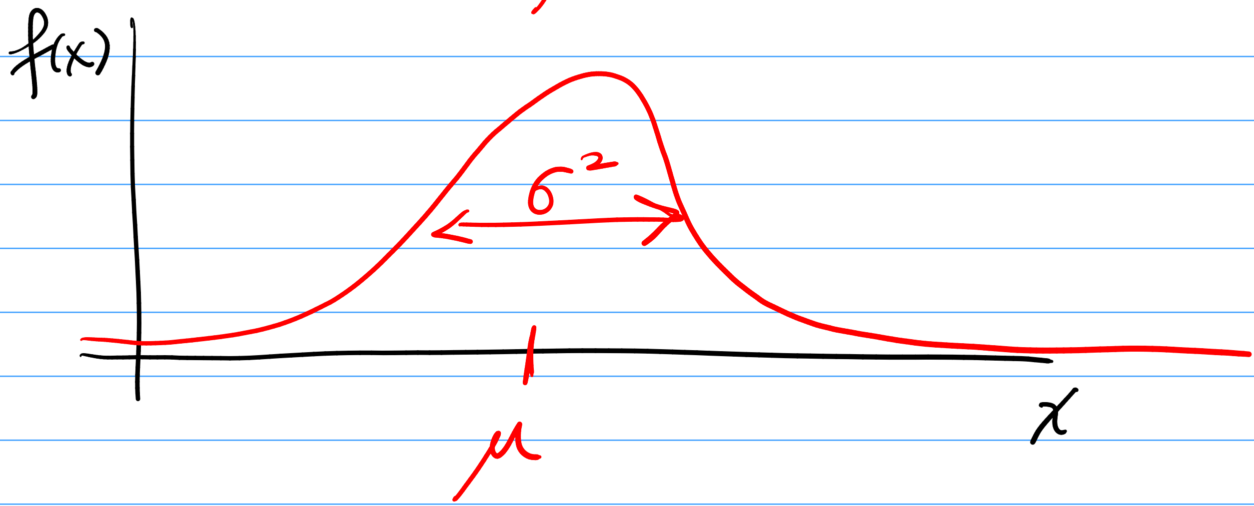
$$M(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}.$$

Normal Distribution

$$X \sim N(\mu, \sigma^2)$$

$\sigma^2 > 0$

$\mu \in \mathbb{R}$



PDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

for $x \in \mathbb{R}$

CDF: no simple formula

Claim: $\mu = E[X]$ and $\sigma^2 = Var(X)$.

MGF:

$$M(t) = E[e^{tX}]$$

$$= \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx$$

exponent:

$$tx - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$= tx - \frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2)$$

$$= -\frac{1}{2\sigma^2}(-2\sigma^2 tx + x^2 - 2\mu x + \mu^2)$$

$$= -\frac{1}{2\sigma^2}\left(\underbrace{x^2 - 2x(\sigma^2 t + \mu)}\, + \mu^2\right)$$

are first two terms in

$$(x-(\mu+\sigma^2 t))^2$$

$$= -\frac{1}{2\sigma^2}\left(x^2 - 2x(\mu+\sigma^2 t) + (\mu+\sigma^2 t)^2 - (\mu+\sigma^2 t)^2 + \mu^2\right)$$

$$= -\frac{1}{2\sigma^2}\left[(x-(\mu+\sigma^2 t))^2 - (\mu+\sigma^2 t)^2 + \mu^2\right]$$

$$M(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(\quad)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2\right) \cdot \underbrace{\exp\left(-\frac{1}{2\sigma^2}\left[-(\mu+\sigma^2 t)^2 + \mu^2\right]\right)}_{\text{no } x} dx$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left[-(\mu+\sigma^2 t)^2 + \mu^2\right]\right) \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2\right) dx}_{=1}$$

PDF of a $N(\mu+\sigma^2 t, \sigma^2)$

$$= \exp\left(-\frac{1}{2\sigma^2}\left[-(\mu+\sigma^2 t)^2+\mu^2\right]\right)$$

$$= \cdots$$

$$= \boxed{\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) = M(t)}$$

$$E[X] = \left.\frac{dM}{dt}\right|_{t=0} = \left.\left(\mu + \frac{\cancel{2}\sigma^2 t}{\cancel{2}}\right)\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\right|_{t=0}$$

$$= (\mu + 0)\exp(0)$$

$$= \mu$$

$$E[X^2] = \left.\frac{d^2 M}{dt^2}\right|_{t=0}$$

$$= \sigma^2 \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) + (\mu+\sigma^2 t)(\mu+\sigma^2 t) \cdot \left.\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\right|_{t=0}$$

$$= \sigma^2 e^0 + (\mu + 0)(\mu + 0) e^0$$

$$= \sigma^2 + \mu^2$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$= (\sigma^2 + \mu^2) - \mu^2$$

$$= \sigma^2.$$

Theorem: Linear Transf of Normal

Let $X \sim N(\mu, \sigma^2)$ and

$$Y = aX + b$$

then $Y \sim N(\underline{a\mu + b}, \underline{a^2\sigma^2})$

$$E[Y] = E[aX + b] = a\,E[X] + b = a\mu + b$$

$$\mathrm{Var}(Y) = \mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X) = a^2\sigma^2.$$

pf. Recall: (1) $M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$

(2) $M_{aX+b}(t) = e^{bt} M_X(at)$

$$M_Y(t) = e^{tb} M_X(at)$$

$$= e^{tb} \exp\left(\mu(at) + \frac{\sigma^2 (at)^2}{2}\right)$$

$$= \exp\left((a\mu + b)t + \frac{a^2\sigma^2 t^2}{2}\right)$$

this is MGF of $N(a\mu + b, a^2\sigma^2)$.