Lecture 12 : Common Dists

Discrete Uniform:

$$Y \sim U(\{a, \dots, b\})$$

if $n = b - a + 1$

then there is some $X \sim U(\{1, \dots, n\})$

so that

$$\underbrace{Y = X + a - 1}_{\text{linear transf.}}.$$

PMF:

$$f(y) = \frac{1}{n} = \frac{1}{b-a+1} \quad \text{for } y = a, a+1, \dots, b$$

Expected Value:

$$E[Y] = E[X + a - 1]$$

$$= E[X] + a - 1$$

$$= \frac{1+n}{2} + a - 1$$

$$= \frac{1+b-a+1}{2} + a - 1 = \frac{a+b}{2}$$

Variance:

$$Var(Y) = Var(X + a - 1)$$
$$= Var(X)$$
$$= \frac{n^2 - 1}{12}$$
$$= \frac{(b-a+1)^2 - 1}{12}$$

MGF: $\boxed{M_{cX+d}(t) = e^{td} M_X(ct)}$

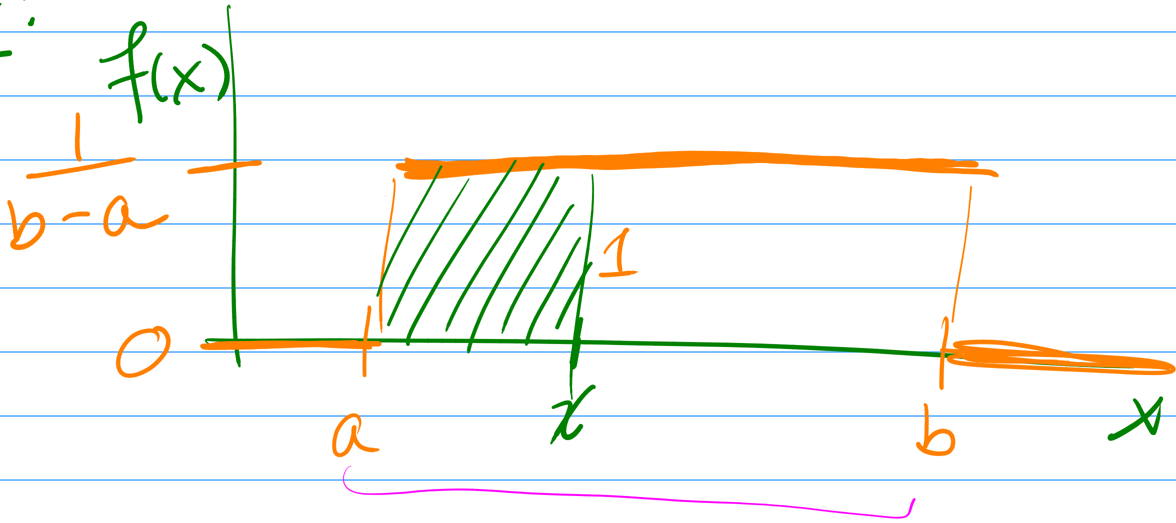
$$M_Y(t) = M_{X+a-1}(t)$$
$$= e^{t(a-1)} M_X(t)$$
$$= e^{t(a-1)} \frac{e^t - e^{t(n+1)}}{n(1-e^t)}$$

$$\boxed{M(t) = e^{t(a-1)} \frac{e^t - e^{t(b-a+2)}}{(b-a+1)(1-e^t)}.}$$

Continuous Uniform

$X \sim U(a, b)$

PDF:



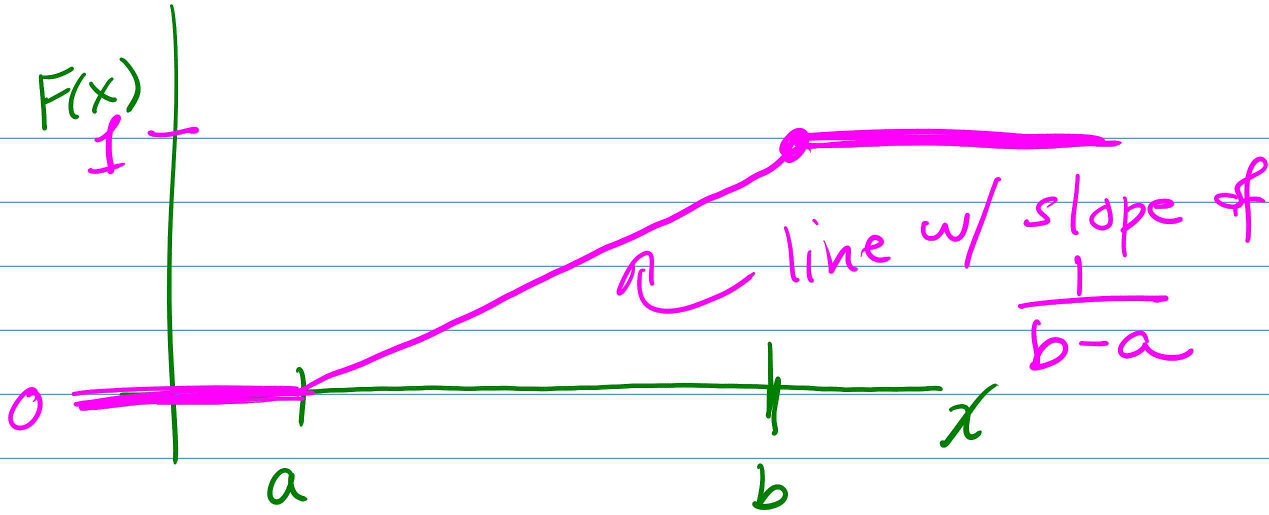
$$f(x) = \frac{1}{b-a} \quad \text{for } a < x < b.$$

CDF:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{a}^{x} \frac{1}{b-a}\,dt$$

$a < x < b$

$$= \frac{t}{b-a}\Big|_{a}^{x} = \frac{x-a}{b-a}$$



Expected Value

$$E[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} \frac{x^2}{2} \Big|_a^b$$

$$\boxed{E[g(X)] = \int g(x) f(x)\,dx}$$

$$= \frac{1}{b-a} \frac{1}{2} (b^2 - a^2)$$

$$= \frac{(a+b)\cancel{(b-a)}}{2\cancel{(b-a)}} = \frac{a+b}{2}.$$

$$E[X^2] = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{x^3}{3(b-a)} \Big|_a^b$$

$$= \frac{b^3 - a^3}{3(b-a)} = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{b^2+ab+a^2}{3}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= \left(\frac{b^2+ab+a^2}{3}\right) - \left(\frac{a+b}{2}\right)^2$$

$$= \dots$$

$$= \frac{(b-a)^2}{12}.$$

$$SD(X) = \sqrt{Var(X)} = |b-a|/\sqrt{12}$$

MGF:

$$M(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\,\frac{1}{t}\,e^{tx}\Big|_a^b$$

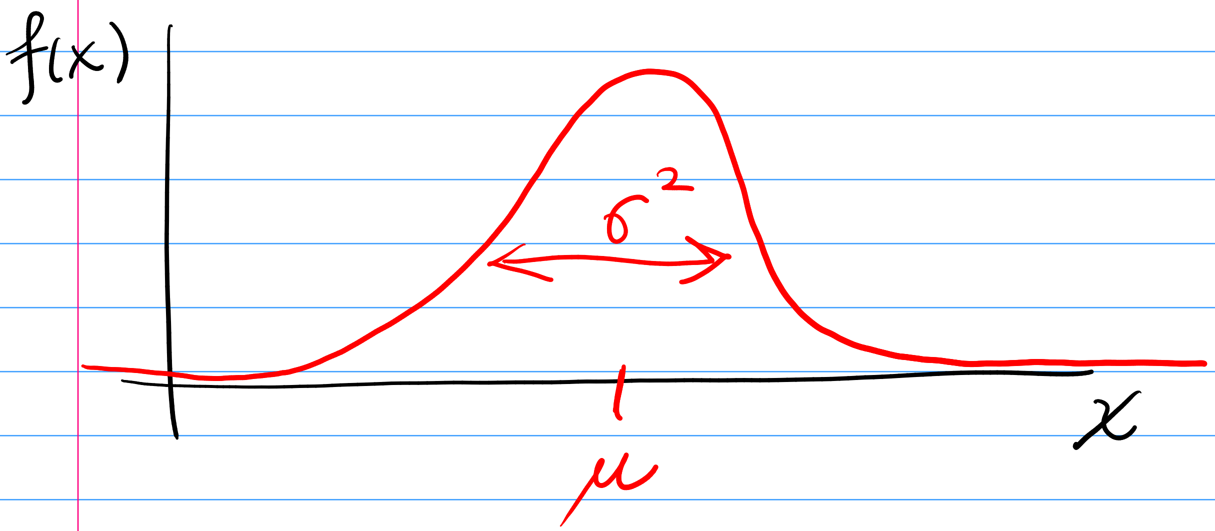
$$\boxed{M(t) = \frac{e^{tb}-e^{ta}}{t(b-a)}}$$

Normal Dist

$$X \sim N(\mu, \sigma^2)$$

$\sigma^2 > 0$

$\mu \in \mathbb{R}$



PDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

for $x \in \mathbb{R}$

<u>CDF</u>: no simple formula : $\Phi(x)$

$$\boxed{\underline{\text{Claim}}: \mu = E[X], \ \sigma^2 = Var(X)}$$

<u>MGF:</u>

$$M(t) = E[e^{tx}] = \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

$$= \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx$$

$$\boxed{\exp(a) = e^a}$$

exponent: $tx - \frac{1}{2\sigma^2}(x-\mu)^2$

$$= tx - \frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2)$$

$$= -\frac{1}{2\sigma^2}\left[-2\sigma^2 tx + x^2 - 2\mu x + \mu^2\right]$$

$$= -\frac{1}{2\sigma^2}\Big[\underbrace{x^2 - 2x(\mu+\sigma^2 t)}_{\text{first two terms in } (x-(\mu+\sigma^2 t))^2} + \mu^2\Big]$$

$$= -\frac{1}{2\sigma^2}\Big[\boxed{x^2 - 2x(\mu+\sigma^2 t) + (\mu+\sigma^2 t)^2} - (\mu+\sigma^2 t)^2 + \mu^2\Big]$$

$$= -\frac{1}{2\sigma^2}\Big[(x-(\mu+\sigma^2 t))^2 - (\mu+\sigma^2 t)^2 + \mu^2\Big]$$

$$M(t) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(\;)\,dx$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2\right) \underbrace{\exp\left(\frac{-(\mu+\sigma^2 t)^2+\mu^2}{-2\sigma^2}\right)}_{\text{no } x} dx$$

$$= \exp\left(\frac{-(\mu+\sigma^2 t)^2+\mu^2}{-2\sigma^2}\right) \overbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2\right) dx}^{1}$$

PDF of a $N(\mu+\sigma^2 t, \sigma^2)$

so integral is 1.

$$M(t) = \exp\left(\frac{-(\mu+\sigma^2 t)^2 + \mu^2}{-2\sigma^2}\right)$$

$$\boxed{= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)}$$

$$E[X] = \left.\frac{dM}{dt}\right|_{t=0} = \left.\left(\mu + \frac{\cancel{2}\sigma^2 t}{\cancel{2}}\right)\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\right|_{t=0}$$

$$= (\mu + 0)(e^0)$$

$$= \mu.$$

$$E[X^2] = \left.\frac{d^2M}{dt^2}\right|_{t=0} = (\sigma^2)\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

$$+ (\mu+\sigma^2 t)(\mu+\sigma^2 t)\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

plug in $t=0$

$$= \sigma^2 e^0 + (\mu+0)(\mu+0)e^0$$

$$= \sigma^2 + \mu^2$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= (\sigma^2+\mu^2) - \mu^2 = \sigma^2$$

Theorem: Linear Transf and Normal

Let $X \sim N(\mu, \sigma^2)$ and

$$Y = aX + b$$

then

$$\boxed{Y \sim N(a\mu+b,\ a^2\sigma^2)}$$

$$E[Y] = E[aX+b] = aE[X]+b = a\mu+b$$

$$Var(Y) = Var(aX+b) = a^2 Var(X) = a^2\sigma^2.$$

pf. Recall: (1) $M_X(t) = \exp(\mu t + \sigma^2 t^2/2)$

(2) $M_{aX+b}(t) = e^{tb} M_X(at)$

$$M_Y(t) = e^{tb} M_X(at)$$

$$= e^{tb} \exp\left(\mu(at) + \sigma^2 (at)^2/2\right)$$

$$= \underbrace{\boxed{\exp\left((a\mu+b)t + (a^2\sigma^2)t^2/2\right)}}_{\text{MGF of } N(a\mu+b,\ a^2\sigma^2)}$$