Lecture 13:

Poisson Distribution

- discrete RV
- support $\{0, 1, 2, 3, ...\}$

Canonical Experiment

Count the number of events that happen in some time/space period.

Ex. - Count # fish in river in hr
- Count # mRNA molecules in cell
- radioactive decay

$X \sim Pois(\lambda)$

$\lambda > 0$, rate of occurrence

\# events

PMF: $f(x) = \dfrac{e^{-\lambda} \lambda^x}{x!}$ for $x = 0, 1, 2, 3, ...$

Expected Value

$$E[X] = \sum_x x f(x) = \sum_{x=\cancel{0}1}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x+1}}{x!}$$

$$= \lambda e^{-\lambda} \boxed{\sum_{x=0}^{\infty} \frac{\lambda^x}{x!}}$$

$$e^y = \sum_{i=0}^{\infty} \frac{y^i}{i!}$$

Taylor series

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$\boxed{E[X] = \lambda}$$

$$E[g(X)] = \sum_x g(x) f(x)$$

$$E[X(X-1)] = \sum_{x=\cancel{0}2}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-2)!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^{x+2}}{x!}$$

$$\frac{x(x-1)}{x!} = \frac{\cancel{x(x-1)}}{\cancel{x(x-1)}(x-2)!}$$

$$= e^{-\lambda} \lambda^2 \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} \lambda^2 e^{\lambda}$$

$$\boxed{E[X(X-1)] = \lambda^2}$$

$$\to E[X^2 - X] = E[X^2] - E[X] = \lambda^2$$

So

$$\boxed{\begin{aligned} E[X^2] &= \lambda^2 + E[X] \\ &= \lambda^2 + \lambda \end{aligned}}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= (\lambda^2 + \lambda) - (\lambda)^2$$

$$\boxed{Var(X) = \lambda}$$

MGF:

$$M(t) = E[e^{tX}]$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!}$$

$$= e^{-\lambda} \exp(\lambda e^{t})$$

$$\boxed{= \exp(\lambda(e^{t}-1)) = M(t)}$$

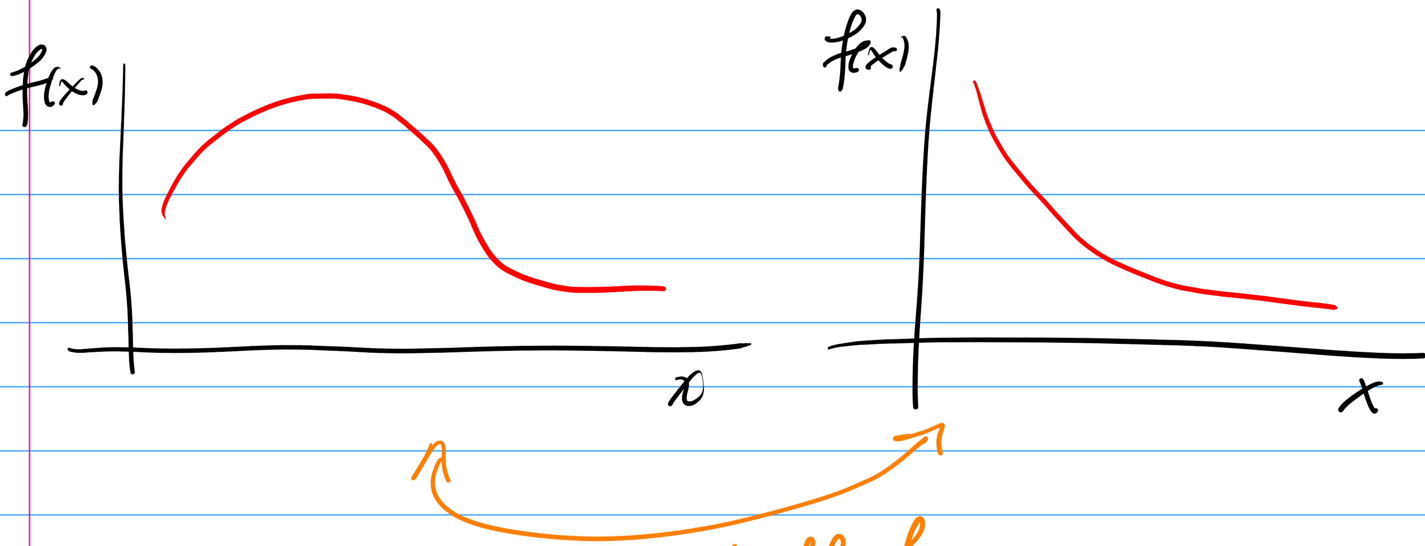
Gamma Dist

- cts dis w/ support $(0,\infty)$
- generalization of $\operatorname{Exp}(\lambda)$

$X \sim \operatorname{Gamma}(k, \lambda)$

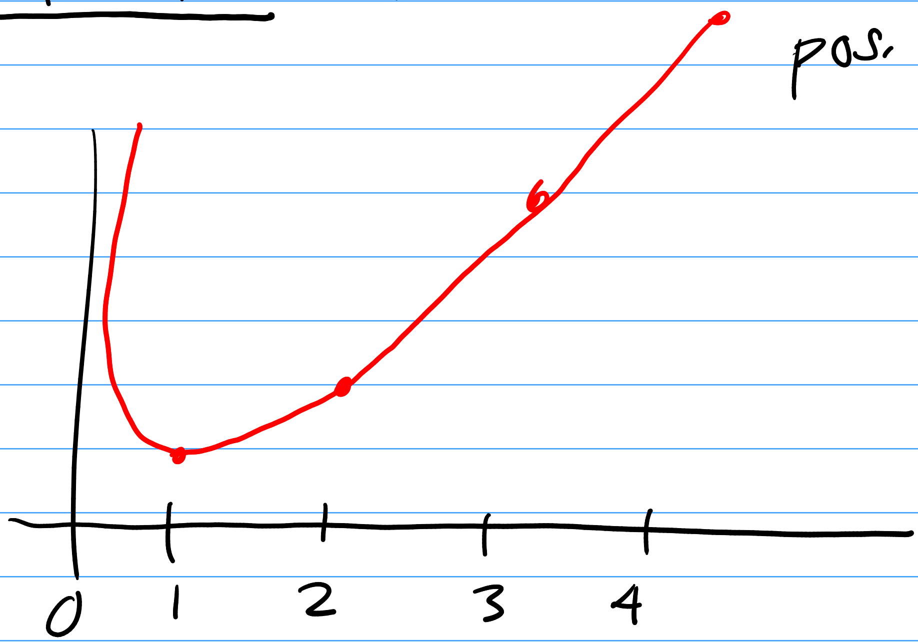
shape $k>0$

rate $\lambda>0$



same λ, diff k

Gamma Function : Extend Factorial to pos. real.



$\Gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$

For k we define

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} \, dx$$

Properties:

$$\boxed{\begin{array}{l}\text{(1) If } k>0 \text{ is an integer then} \\ \Gamma(k) = (k-1)! \\ \Gamma(k+1) = k! \end{array}}$$

Note: $\Gamma(k) = (k-1)! = (k-1)(k-2)!$

$= (k-1)\Gamma(k-1)$

alt: $\Gamma(k+1) = k\Gamma(k)$

$$\boxed{\begin{array}{l}\text{(2) This is true for any } k>1 \\ \text{(i) } \Gamma(k) = (k-1)\Gamma(k-1) \\ \text{(ii) } \Gamma(k+1) = k\Gamma(k) \end{array}}$$

Trivia: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$X \sim \text{Gamma}(k, \lambda)$$

<u>PDF:</u>

$$f(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{k-1}}{\Gamma(k)}, \quad x > 0$$

<u>Note:</u> $k = 1$ then $X \sim \text{Exp}(\lambda)$

<u>Expected Value:</u>

$$E[X] = \int_{\mathbb{R}} x f(x)\, dx$$

$$= \int_0^\infty x \frac{\lambda e^{-\lambda x} (\lambda x)^{k-1}}{\Gamma(k)}\, dx$$

Looks like PDF of Gamma$(k+1, \lambda)$

$$\frac{\lambda e^{-\lambda x} (\lambda x)^{k}}{\Gamma(k+1)}$$

$$= \underbrace{\frac{\Gamma(k+1)}{\Gamma(k)\lambda} \underbrace{\int_0^\infty \frac{\lambda e^{-\lambda x} x^{k} \lambda^{k-1} \lambda}{\Gamma(k+1)}\, dx}_{\text{identical}}}_{1}$$

$$= \frac{\Gamma(k+1)}{\Gamma(k)\,\lambda}$$

$$= \frac{k\,\cancel{\Gamma(k)}}{\cancel{\Gamma(k)}\,\lambda} = \boxed{\frac{k}{\lambda} = E[X]}$$

$$E[X^r] = \int_0^\infty x^r \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)}\,dx$$

looks like PDF

$$\text{Gamma}(k+r, \lambda) \quad \frac{\lambda e^{-\lambda x}(\lambda x)^{k+r-1}}{\Gamma(k+r)}$$

$$= \frac{\Gamma(k+r)}{\Gamma(k)\lambda^r}\underbrace{\int_0^\infty \frac{\lambda e^{-\lambda x}\, x^{k+r-1}\lambda^{k-1}\lambda^r}{\Gamma(k+r)}\,dx}_{\text{integrates to } 1}$$

$$\boxed{E[X^r] = \frac{\Gamma(k+r)}{\Gamma(k)\,\lambda^r}}$$

$$E[X^2] = \frac{\Gamma(k+2)}{\Gamma(k)}\frac{1}{\lambda^2}$$

$$= \frac{(k+1)\Gamma(k+1)}{\Gamma(k)}\frac{1}{\lambda^2} = \frac{(k+1)k\,\cancel{\Gamma(k)}}{\cancel{\Gamma(k)}}\frac{1}{\lambda^2}$$

$$E[X^2] = \frac{k(k+1)}{\lambda^2}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{k(k+1)}{\lambda^2} - \left(\frac{k}{\lambda}\right)^2$$

$$= k/\lambda^2 .$$

Geometric Dist

Canonical Experiment:

Flip coins (indep) each w/ a prob. p of Hs, until I get my first H.

X = # flips until first H

Support: 1, 2, 3, 4, ...

$X \sim Geom(p)$

PMF: $f(x) = (1-p)^{x-1} p$ for $x = 1, 2, \ldots$

$$F(x) = 1 - (1-p)^{\lfloor x \rfloor} \text{ for } x \geq 1$$

Recall: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$ for $|r| < 1$

Expected Value

$$E[X] = \sum_{x=1}^{\infty} x (1-p)^{x-1} p$$

$$= p \sum_{x=1}^{\infty} \underbrace{x (1-p)^{x-1}}$$

looks like $-\frac{d}{dp}(1-p)^x$

$$= -p \sum_{x=1}^{\infty} \frac{d}{dp}(1-p)^x$$

$$= -p \frac{d}{dp} \sum_{x=1}^{\infty} (1-p)^x$$

basically geometric

$$= -p \frac{d}{dp} \left[\sum_{x=0}^{\infty} (1-p)^{x+1} \right]$$

$$= -p \frac{d}{dp} \left[(1-p) \sum_{x=0}^{\infty} (1-p)^x \right]$$

$$\frac{1}{1-(1-p)} = \frac{1}{p}$$

$$= -p \frac{d}{dp}\left[(1-p)/p\right]$$

$$= -p\left(-1/p^2\right)$$

$$\boxed{E[X] = \frac{1}{p}}$$