# Lecture 13

## Poisson Dist

- discrete
- support : 0, 1, 2, 3, 4, ...

## Canonical Experiment:

Counting number of occurrences of events that happen in some time/space interval.

Ex. - count # fish in river in 1 hr

- count # mRNA molecules in cell

- radioactive decay

$X \sim \text{Pois}(\lambda)$

$\lambda > 0$, rate of events

\# events

PMF: $$f(x) = \frac{e^{-\lambda} \lambda^{x}}{x!} \text{ for } x = 0, 1, 2, 3, \ldots$$

# Expected Value:

$$E[X] = \sum_x x f(x)$$

$$= \sum_{x=\cancel{0}1}^{\infty} \frac{x e^{-\lambda} \lambda^x}{x!}$$

$$\frac{x}{x!} = \frac{\cancel{x}}{\cancel{x}(x-1)!} = \frac{1}{(x-1)!}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^{x}}{(x-1)!}$$

Recall:

$$e^y = \sum_{i=0}^{\infty} y^i / i!$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x+1}}{x!}$$

$$= \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$\boxed{E[X] = \lambda}$$

$$E[X(X-1)] = \sum_{x=\cancel{0}2}^{\infty} \frac{x(x-1) e^{-\lambda} \lambda^x}{x!}$$

$$\frac{x(x-1)}{x!} = \frac{\cancel{x(x-1)}}{\cancel{x(x-1)}(x-2)!} = \frac{1}{(x-2)!}$$

$$= e^{-\lambda} \sum_{x=2} \frac{\lambda^x}{(x-2)!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x+2}}{x!}$$

$$= \lambda^2 e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}}_{e^{\lambda}}$$

$$E[X(X-1)] = \lambda^2 e^{-\lambda} e^{\lambda} = \lambda^2$$

$$E[X^2 - X] = E[X^2] - E[X] = \lambda^2$$

so $E[X^2] = \lambda^2 + E[X] = \lambda^2 + \lambda$

$$Var(X) = E[X^2] - E[X]^2$$

$$= \lambda^2 + \lambda - (\lambda)^2$$

$$= \lambda$$

<u>MGF:</u>

$$M(t) = E[e^{tX}]$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} (\lambda e^t)^x / x!}_{\exp(\lambda e^t)}$$

$$= e^{-\lambda} \exp(\lambda e^t)$$

$$\boxed{M(t) = \exp(\lambda(e^t - 1))}$$

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**Exponential Dist:**

$f(x) = \lambda e^{-\lambda x}$ for $x > 0$, $\lambda > 0$

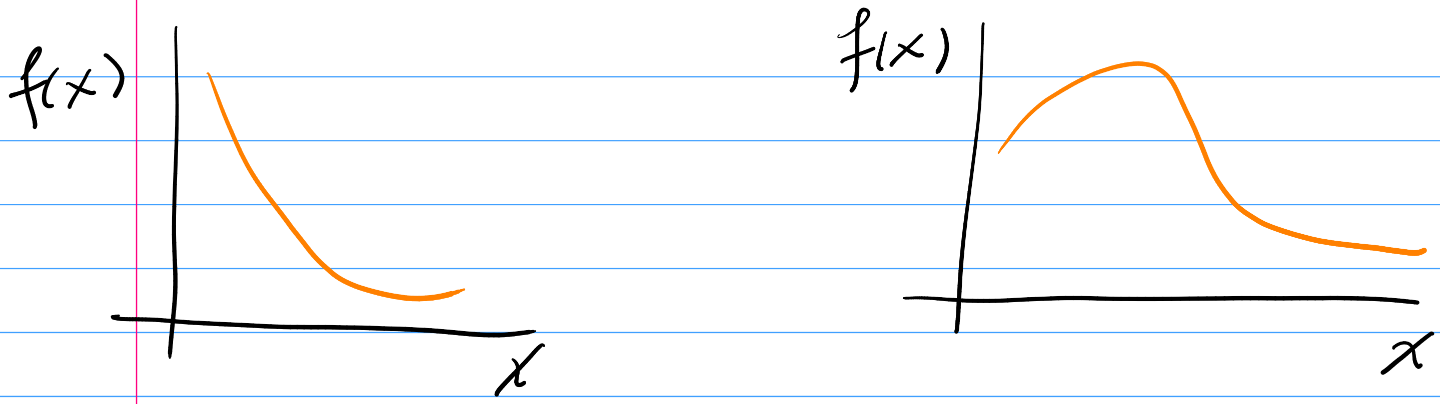
**Gamma Dist:**

- cts dist w/ support $(0, \infty)$
- generalization of $\text{Exp}(\lambda)$

$X \sim \text{Gamma}(k, \lambda)$

$\lambda > 0$, rate

$k > 0$, shape




same $\lambda$, diffrent shape

**Gamma Function**: Extend factorials to pos numbers


For $k > 0$ we define

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} \, dx$$

## Properties

① If $k>0$ is an integer then

$$\Gamma(k) = (k-1)!$$

$$\Gamma(k+1) = k!$$

Note: $\Gamma(k) = (k-1)! = (k-1)(k-2)!$

$$= (k-1)\Gamma(k-1)$$

$$\Gamma(k+1) = k\Gamma(k)$$

② This is true for all values of $k>1$:

(i) $\Gamma(k) = (k-1)\Gamma(k-1)$

(ii) $\Gamma(k+1) = k\Gamma(k)$

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Let $X \sim \text{Gamma}(k, \lambda)$

PDF:

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)}, \quad x>0$$

note: If $k=1$ then $X \sim \text{Exp}(\lambda)$.

## Expected Value:

$$c \underbrace{\int PDF}_{1} = c$$

$$E[X] = \int_0^\infty x \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)}\,dx$$

looks like Gamma$(k+1, \lambda)$

$$\frac{\lambda e^{-\lambda x}(\lambda x)^{k}}{\Gamma(k+1)}$$

$$= \frac{\Gamma(k+1)}{\Gamma(k)\lambda} \underbrace{\int_0^\infty \frac{\lambda e^{-\lambda x} \lambda^{k-1} x^{k} \lambda \, dx}{\Gamma(k+1)}}_{1}$$

$$= \frac{\Gamma(k+1)}{\Gamma(k)\lambda} = \frac{k\cancel{\Gamma(k)}}{\cancel{\Gamma(k)}\lambda} = \boxed{\frac{k}{\lambda} = E[X]}$$

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$$E[X^r] = \int_0^\infty x^r \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)}\,dx$$

$$= \frac{\Gamma(k+r)}{\lambda^r \Gamma(k)} \underbrace{\int_0^\infty \frac{\lambda e^{-\lambda x} \lambda^{k-1} x^{k+r-1} \lambda^r}{\Gamma(k+r)}\,dx}_{1}$$

looks like: Gamma$(k+r, \lambda)$

$$\frac{\lambda e^{-\lambda x}(\lambda x)^{k+r-1}}{\Gamma(k+r)}$$

$$= \boxed{\frac{\Gamma(k+r)}{\lambda^r \Gamma(k)} = E[X^r]}$$

$$E[X^2] = \frac{\Gamma(k+2)}{\lambda^2\Gamma(k)} = \frac{(k+1)\Gamma(k+1)}{\lambda^2\Gamma(k)}$$

$$= \frac{(k+1)k\cancel{\Gamma(k)}}{\lambda^2\cancel{\Gamma(k)}}$$

$$= \frac{(k+1)k}{\lambda^2}$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$= \frac{(k+1)k}{\lambda^2} - \left(\frac{k}{\lambda}\right)^2$$

$$= \ldots$$

$$= k/\lambda^2.$$

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## Geometric Distribution

**Canonical Experiment:**

Flip a coin repeatedly (indep) until I get my first Hs – prob. of getting H on any flip is $p$.

$X$ = # flips I make

Support: 1, 2, 3, 4, ...

$X \sim Geom(p)$

PMF: $f(x) = (1-p)^{x-1} p$ for $x = 1, 2, 3, \ldots$

CDF: $F(x) = 1 - ((1-p)^{\lfloor x \rfloor}$, for $x \geq 1$

Recall: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$ for $|r| < 1$

Expected Value:

look like: $-\frac{d}{dp}(1-p)^x$

$$E[X] = \sum_{x=1}^{\infty} x(1-p)^{x-1} p$$

$$= -p \sum_{x=1}^{\infty} \frac{d}{dp}(1-p)^x$$

$$= -p \frac{d}{dp}\left[\sum_{x=1}^{\infty}(1-p)^x\right]$$

$$= -p \frac{d}{dp}\left[\sum_{x=0}^{\infty}(1-p)^{x+1}\right]$$

$$= -p \frac{d}{dp}\left[(1-p)\underbrace{\sum_{x=0}^{\infty}(1-p)^{x}}_{\frac{1}{1-(1-p)}}\right]$$

$$= -p \frac{d}{dp}\left[(1-p)/p\right]$$

$$= -p\left(-1/p^2\right)$$

$$= \boxed{E[X] = 1/p}$$