Lecture 14

$X \sim \text{Geom}(p)$ $\quad p \in [0,1]$

$f(x) = (1-p)^{x-1} p$ for $x = 1, 2, 3, \ldots$

MGF:

$$M(t) = E[e^{tX}]$$

$$= \sum_{x=1}^{\infty} e^{tx} (1-p)^{x-1} p$$

$$= \frac{p}{1-p} \sum_{x=1}^{\infty} \left((1-p)e^{t}\right)^{x}$$

$$= \frac{p}{1-p} \sum_{x=0}^{\infty} \left((1-p)e^{t}\right)^{x+1}$$

$$= \frac{(1-p)e^{t} p}{1-p} \underbrace{\sum_{x=0}^{\infty} \left(\underbrace{(1-p)e^{t}}_{r}\right)^{x}}_{1/1-r} \qquad |r| < 1$$

$$\boxed{M(t) = pe^{t} \frac{1}{1-(1-p)e^{t}}} \text{ for } \underbrace{(1-p)e^{t} < 1}_{t < -\log(1-p)}$$

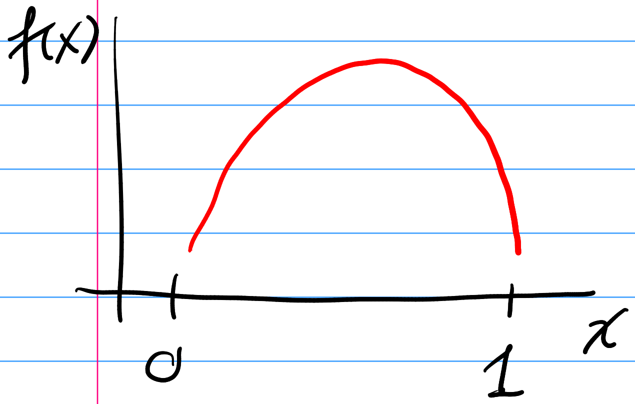
$$E[X] = 1/p = \frac{dM}{dt}\Big|_{t=0}$$

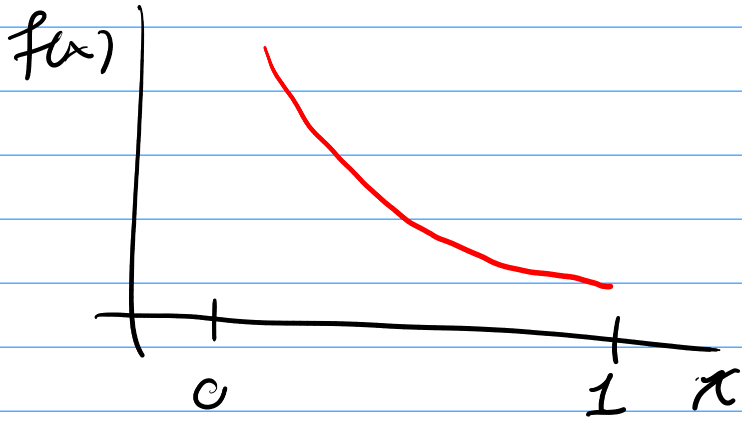
$$\frac{d^2M}{dt^2}\Big|_{t=0} = E[X^2] = \frac{2-p}{p^2}.$$

$$\begin{aligned} Var(X) &= E[X^2] - E[X]^2 \\ &= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 \\ &= \frac{1-p}{p^2}. \end{aligned}$$

Beta Distribution

- continuous dist on [0, 1]





Beta Function:

$$B : \mathbb{R}^2 \to \mathbb{R}$$

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

PDF:

$X \sim \text{Beta}(a,b)$, $a, b > 0$

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \quad \text{for } 0 < x < 1$$

$$E[X^r] = \int_0^1 \frac{x^r x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

$$= \frac{B(a+r,b)}{B(a,b)} \underbrace{\int_0^1 \frac{x^{a+r-1}(1-x)^{b-1}}{B(a+r,b)}\,dx}_{1}$$

claim: looks like PDF of Beta(a+r, b)

$$\boxed{\frac{x^{a+r-1}(1-x)^{b-1}}{B(a+r,b)}}$$

$$= \boxed{\frac{B(a+r,b)}{B(a,b)} = E[X^r]}$$

$$E[X] = \frac{B(a+1,b)}{B(a,b)} = \frac{\dfrac{\Gamma(a+1)\cancel{\Gamma(b)}}{\Gamma(a+b+1)}}{\dfrac{\Gamma(a)\cancel{\Gamma(b)}}{\Gamma(a+b)}}$$

$$= \frac{\Gamma(a+1)}{\Gamma(a)} \frac{\Gamma(a+b)}{\Gamma(a+b+1)}$$

$$= \frac{a\cancel{\Gamma(a)}}{\cancel{\Gamma(a)}} \frac{\cancel{\Gamma(a+b)}}{(a+b)\cancel{\Gamma(a+b)}}$$

$$= \frac{a}{a+b}$$

$$E[X^2] = \frac{B(a+2,b)}{B(a,b)} = \frac{\dfrac{\Gamma(a+2)\cancel{\Gamma(b)}}{\Gamma(a+b+2)}}{\dfrac{\Gamma(a)\cancel{\Gamma(b)}}{\Gamma(a+b)}}$$

$$= \frac{\Gamma(a+2)}{\Gamma(a)} \frac{\Gamma(a+b)}{\Gamma(a+b+2)}$$

$$= \frac{(a+1)a\cancel{\Gamma(a)}}{\cancel{\Gamma(a)}} \frac{\cancel{\Gamma(a+b)}}{(a+b+1)(a+b)\cancel{\Gamma(a+b)}}$$

$$E[X^2] = \frac{a(a+1)}{(a+b)(a+b+1)}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{a(a+1)}{(a+b)(a+b+1)} - \left(\frac{a}{a+b}\right)^2$$

$$= \dots$$

$$= \frac{ab}{(a+b)^2(a+b+1)}$$

EXAM 2

Transformations

If I know something about X

what do I know about $Y = g(X)$?

Discrete RVs

Q: If I know f_X can I get f_Y?

$$f_Y(y) = P(Y=y) = P(g(X)=y)$$

If g is invertible

$$= P(X = g^{-1}(y))$$

$$= f_X(g^{-1}(y))$$

If g isn't invertible then

$$f_Y(y) = P(g(X) = y)$$

$$= P(X \in g^{-1}(y))$$

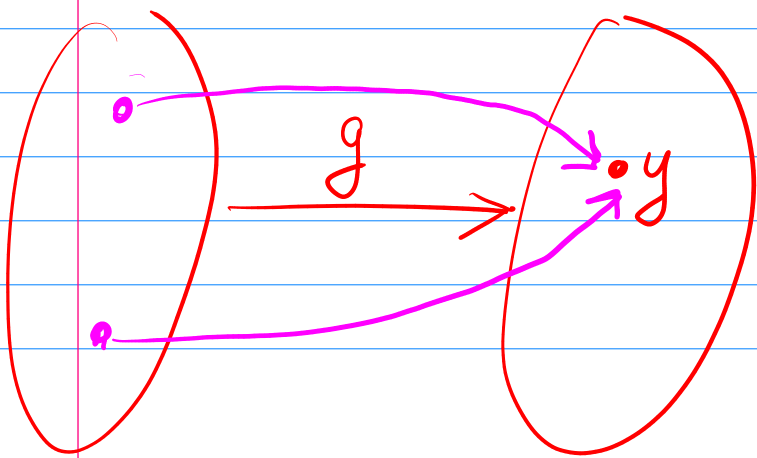
inverse image

$$g^{-1}(y) = \{x : g(x) = y\}$$

$$\boxed{P(X \in A) = \sum_{x \in A} f(x)}$$

$$= \sum_{x \in g^{-1}(y)} f_X(x)$$

$$= \sum_{x : g(x) = y} f_X(x)$$



Theorem: If X is discrete and $Y = g(X)$ then

$$f_Y(y) = \sum_{x: g(x)=y} f_X(x)$$

Ex. Let $X \sim Bin(n, p)$

$$Y = n - X$$

$$y = g(x) = n - x \iff x = n - y = g^{-1}(y)$$

$$f_Y(y) = \sum_{x: g(x)=y} f_X(x)$$

$x: g(x) = y$

$n - x = y$

$x = n - y$

$$= \sum_{x=n-y} f_X(x)$$

$$\boxed{f_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, \ldots, n}$$

$$= f_X(n-y)$$

$$= \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}$$

for $n - y = 0, \ldots, n$

$$= \binom{n}{y} p^{n-y} (1-p)^{y} \quad \text{for } y = 0, \dots, n$$

$q = 1-p$

$$f_Y(y) = \binom{n}{y} q^{y} (1-q)^{n-y} \quad \text{for } y = 0, \dots, n$$

$$Y \sim Bin(n, q)$$

Continuous RVs and CDFs

Theorem: If X is cts and $Y = g(X)$ and g is invertible, then

(1) If g is increasing then

$$F_Y(y) = F_X(g^{-1}(y))$$

(2) If g is decreasing then

$$F_Y(y) = 1 - F_X(g^{-1}(y)).$$

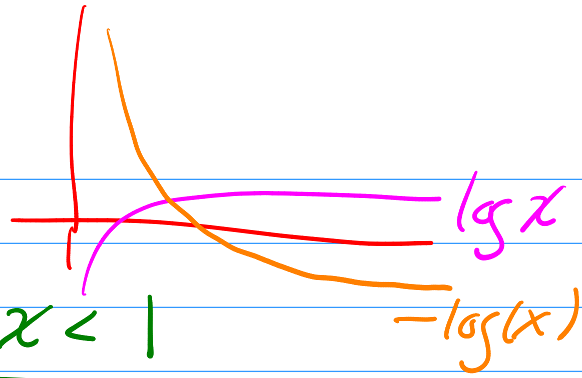
pf. Case 1: g increasing

$$F_Y(y) = P(Y \le y)$$
$$= P(g(X) \le y)$$
$$= P(X \le g^{-1}(y))$$
$$= F_X(g^{-1}(y))$$

$$\boxed{\begin{array}{l} x < y \\ \frac{1}{x} > \frac{1}{y} \\ \log(x) < \log(y) \end{array}}$$

Case 2: g decreasing

$$F_Y(y) = P(Y \le y)$$
$$= P(g(X) \le y)$$
$$= P(X \ge g^{-1}(y))$$
$$= 1 - P(X < g^{-1}(y))$$
$$= 1 - F_X(g^{-1}(y))$$



Ex, $X \sim U(0,1)$

$F_X(x) = x$ for $0 < x < 1$

Consider $Y = -\log(X)$

Support of Y is $(0, \infty)$

$y = -\log(x) \Leftrightarrow -y = \log(x)$

$\Leftrightarrow e^{-y} = x = g^{-1}(y)$

Apply theorem:

$F_Y(y) = 1 - F_X(g^{-1}(y))$

$= 1 - F_X(\underbrace{e^{-y}}_{0<e^{-y}<1})$

$0 < X < 1$ then

$\log(x) < 0$

$-\log(x) > 0$

$y =$

$0 < e^{-y} = \frac{1}{e^{y}} \leq 1$

$= 1 - e^{-y}$

for $y > 0$

CDF of $\text{Exp}(1)$

So $Y \sim \text{Exp}(1)$.

What about PDFs?

Theorem: If X is cts and $Y = g(X)$

and ① g is invertible

② g^{-1} is diff'able

then

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$
