Lecture 14

$X \sim \text{Geom}(p)$ $p \in [0,1]$

$$M(t) = E[e^{tX}]$$
$$= \sum_{x=1}^{\infty} e^{tx} (1-p)^{x-1} p$$
$$= \sum_{x=0}^{\infty} e^{t(x+1)} (1-p)^{x} p$$
$$= pe^t \underbrace{\sum_{x=0}^{\infty} (\underbrace{(1-p)e^t}_{r})^x = \frac{1}{1-r}}_{\text{for } |r|<1}$$
$$= pe^t \frac{1}{1-(1-p)e^t} \quad \text{for } (1-p)e^t < 1$$

$$\boxed{M(t) = \frac{pe^t}{1-(1-p)e^t} \quad \text{for } t < -\log(1-p)}$$

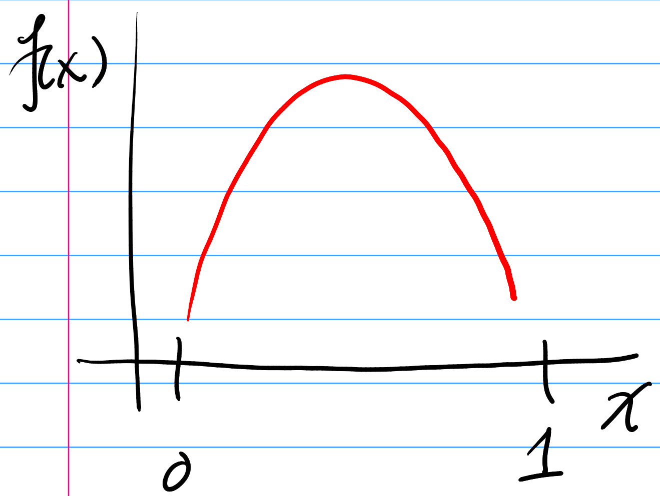
$$E[X] = \left.\frac{dM}{dt}\right|_{t=0} = \cdots = 1/p$$

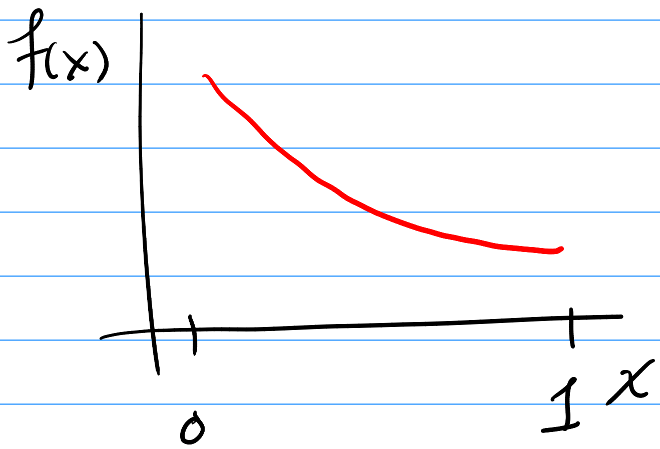
$$E[X^2] = \frac{d^2 M}{dt^2}\bigg|_{t=0} = \cdots = \frac{2-p}{p^2}$$

$$\begin{aligned} Var(X) &= E[X^2] - E[X]^2 \\ &= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 \\ &= \frac{1-p}{p^2}. \end{aligned}$$

Beta Distribution

- cts dist w/ support $[0,1]$





Beta Function $\quad B : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

beta function

Beta dist

PDF: $X \sim \text{Beta}(a,b)$, $a>0, b>0$

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \quad \text{for } 0<x<1$$

$$E[X^r] = \int_0^1 x^r \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

looks like Beta(a+r, b)

$$= \frac{B(a+r,b)}{B(a,b)} \underbrace{\int_0^1 \frac{x^{a+r-1}(1-x)^{b-1}}{B(a+r,b)}\,dx}_{1} \qquad \frac{x^{a+r-1}(1-x)^{b-1}}{B(a+r,b)}$$

$$\Rightarrow \boxed{E[X^r] = \frac{B(a+r,b)}{B(a,b)}}$$

$$E[X] = \frac{B(a+1,b)}{B(a,b)} = \frac{\dfrac{\Gamma(a+1)\cancel{\Gamma(b)}}{\Gamma(a+b+1)}}{\dfrac{\Gamma(a)\cancel{\Gamma(b)}}{\Gamma(a+b)}}$$

$$= \frac{\Gamma(a+1)}{\Gamma(a)} \frac{\Gamma(a+b)}{\Gamma(a+b+1)}$$

$$= \frac{a\cancel{\Gamma(a)}}{\cancel{\Gamma(a)}} \frac{\cancel{\Gamma(a+b)}}{(a+b)\cancel{\Gamma(a+b)}}$$

$$= \frac{a}{a+b}.$$

$$E[X^2] = \frac{B(a+2,b)}{B(a,b)} = \frac{\frac{\Gamma(a+2)\cancel{\Gamma(b)}}{\Gamma(a+b+2)}}{\frac{\Gamma(a)\cancel{\Gamma(b)}}{\Gamma(a+b)}}$$

$$= \frac{\Gamma(a+2)}{\Gamma(a)} \frac{\Gamma(a+b)}{\Gamma(a+b+2)}$$

$$= \frac{(a+1)a\cancel{\Gamma(a)}}{\cancel{\Gamma(a)}} \frac{\cancel{\Gamma(a+b)}}{(a+b+1)(a+b)\cancel{\Gamma(a+b)}}$$

$$= \boxed{\frac{a(a+1)}{(a+b)(a+b+1)} = E[X^2]}$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$= \frac{a(a+1)}{(a+b)(a+b+1)} - \left(\frac{a}{a+b}\right)^2$$

$$= \cdots$$

$$\boxed{= \frac{ab}{(a+b)^2(a+b+1)}}$$

EXAM 2

Transformations

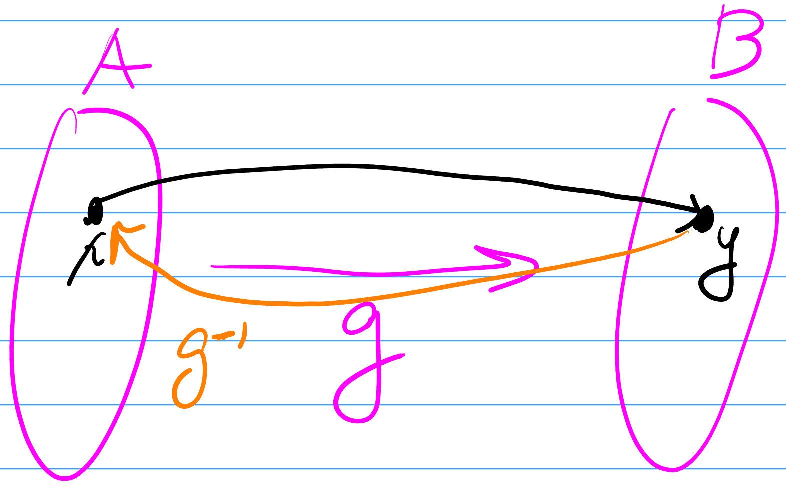
If I know something about X what do I know about $Y = g(X)$?

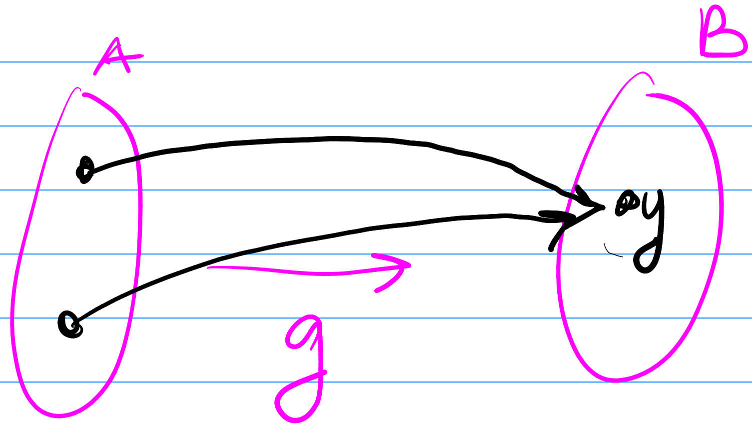
Discrete RVs

Q: If I know f_X can I get f_Y?

Inverses / Inverse Images

Invertible





Inverse Image: $g^{-1}(y) = \{x : g(x) = y\}$

Notice: $f_Y(y) = P(Y = y)$

$$= P(g(X) = y)$$

If g is invertible

$$= P(X = g^{-1}(y))$$

$$= f_X(g^{-1}(y))$$

If g isn't invertible

$$= P(X \in g^{-1}(y))$$

inverse image

$$\boxed{P(X \in A) = \sum_{x \in A} f_X(x)}$$

$$= \sum_{x \in g^{-1}(y)} f_X(x)$$

$$= \sum_{x : g(x) = y} f_X(x)$$

<u>Theorem:</u> If X is discrete and $Y = g(X)$ then

$$f_Y(y) = \sum_{x : g(x) = y} f_X(x)$$

<u>Ex.</u> $X \sim Bin(n, p)$

$$Y = n - X$$

$$y = g(x) = n - x \iff x = n - y = g^{-1}(y)$$

$$f_Y(y) = \sum_{x : g(x) = y} f_X(x) \qquad x = n - y$$

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$
for
$x = 0, \ldots, n$

$$= \sum_{x = n-y} f_X(x)$$

$$= f_X(n-y)$$

$$= \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}$$
for $n-y = 0, \ldots, n$

$$= \binom{n}{y} p^{n-y} (1-p)^{y} \text{ for } y = 0, \ldots, n$$

$q = 1 - p$

$$\underbrace{f_Y(y) = \binom{n}{y} q^{y} (1-q)^{n-y} \text{ for } y = 0, \ldots, n}_{\mathrm{Bin}(n, q)}$$

Continuous RVs and CDF

Theorem: If X is cts and $Y = g(X)$ and g is invertible, then

1. If <u>g is increasing</u> then
$$F_Y(y) = F_X(g^{-1}(y))$$

2. If <u>g is decreasing</u> then
$$F_Y(y) = 1 - F_X(g^{-1}(y)).$$

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pf: Case 1:

$$F_Y(y) = P(Y \le y)$$
$$= P(g(X) \le y)$$
$$= P(X \le g^{-1}(y))$$
$$= F_X(g^{-1}(y))$$

$x < y$

$\log x < \log y$

$x < y$

$\frac{1}{x} > \frac{1}{y}$

Case 2:

$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$
$$= P(X \ge g^{-1}(y))$$
$$= 1 - P(X < g^{-1}(y))$$
$$= 1 - F_X(g^{-1}(y)).$$

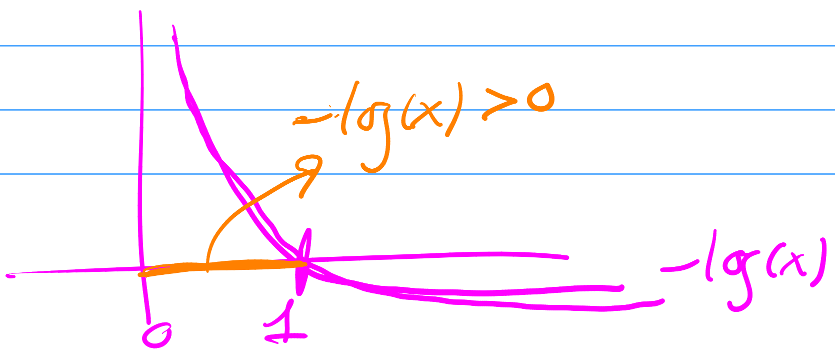
Ex, $X \sim U(0,1)$

$F_X(x) = x$ for $0 < x < 1$

$Y = -\log X$, support of Y is $(0, \infty)$

$$y = -\log(x) \Rightarrow -y = \log x$$
$$\Rightarrow e^{-y} = x = g^{-1}(y)$$

Apply theorem:



$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

$$= 1 - F_X(\underbrace{e^{-y}})$$

btwn 0 and 1

$$\underbrace{F_Y(y) = 1 - e^{-y}}$$

CDF of Exp(1)

$e^{-y} > 0$

$y > 0$ then $e^{-y} = \frac{1}{e^y} \leq 1$

i.e.

$Y \sim Exp(1)$.

$$\boxed{\begin{array}{l}\text{CDF of } z \sim Exp(\lambda) \\ 1 - e^{-\lambda z} \quad \text{for } z > 0\end{array}}$$

What about PDFs?

Theorem: If X is cts and $Y = g(X)$

and (1) g is invertible

(2) g^{-1} is differentiable

then

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$