Lecture 15

Theorem: If X is a cts RV and

$Y = g(X)$ and

1. g is invertible
2. g^{-1} is differentiable

then

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

Pf. **Case 1:** g increasing

prev. CDF thrm said

$$F_Y(y) = F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{dF_Y}{dy} = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

Case 2: g decreasing

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{dF_Y}{dy} = -f_X(g^{-1}(y))\frac{dg^{-1}}{dy}$$

$$= f_X(g^{-1}(y))\left|\frac{dg^{-1}}{dy}\right|$$

Ex. $X \sim \text{Gamma}(k, \lambda)$

$$f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)}, \quad x > 0$$

$Y = 1/X$.

$$g(x) = 1/x = y \Rightarrow x = \boxed{1/y = g^{-1}(y)}$$

$$\boxed{\frac{dg^{-1}}{dy} = -\frac{1}{y^2}}$$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dg^{-1}}{dy}\right|$$

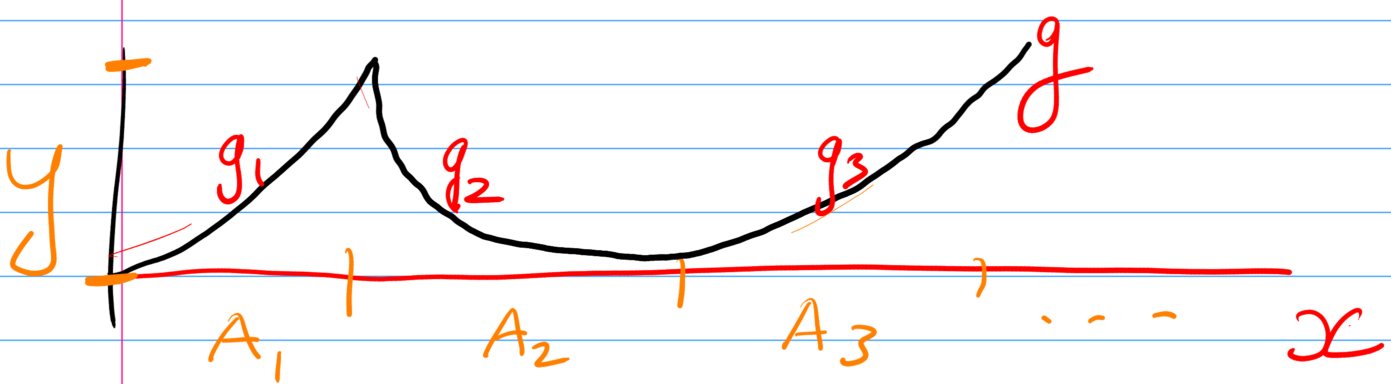
$$= f_X\left(\frac{1}{y}\right)\left|\frac{-1}{y^2}\right|$$

$$\boxed{f_Y(y) = \frac{\lambda e^{-\lambda/y}(\lambda/y)^{k-1}}{\Gamma(k)} \frac{1}{y^2}, \quad y>0}$$

Inverse Gamma dist

What about non-invertible g?

<u>Theorem</u>: Let X be cts w/ support $\mathcal{X}$ and for $i=1,\dots,K$ let A_i partition $\mathcal{X}$



Let g_i be g restricted to A_i. Then if

(1) prev. thrm applies to each g_i

(2) The image of A_i under g_i is the same for all i

then

$$f_Y(y) = \sum_{i=1}^{K} f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}}{dy} \right|$$

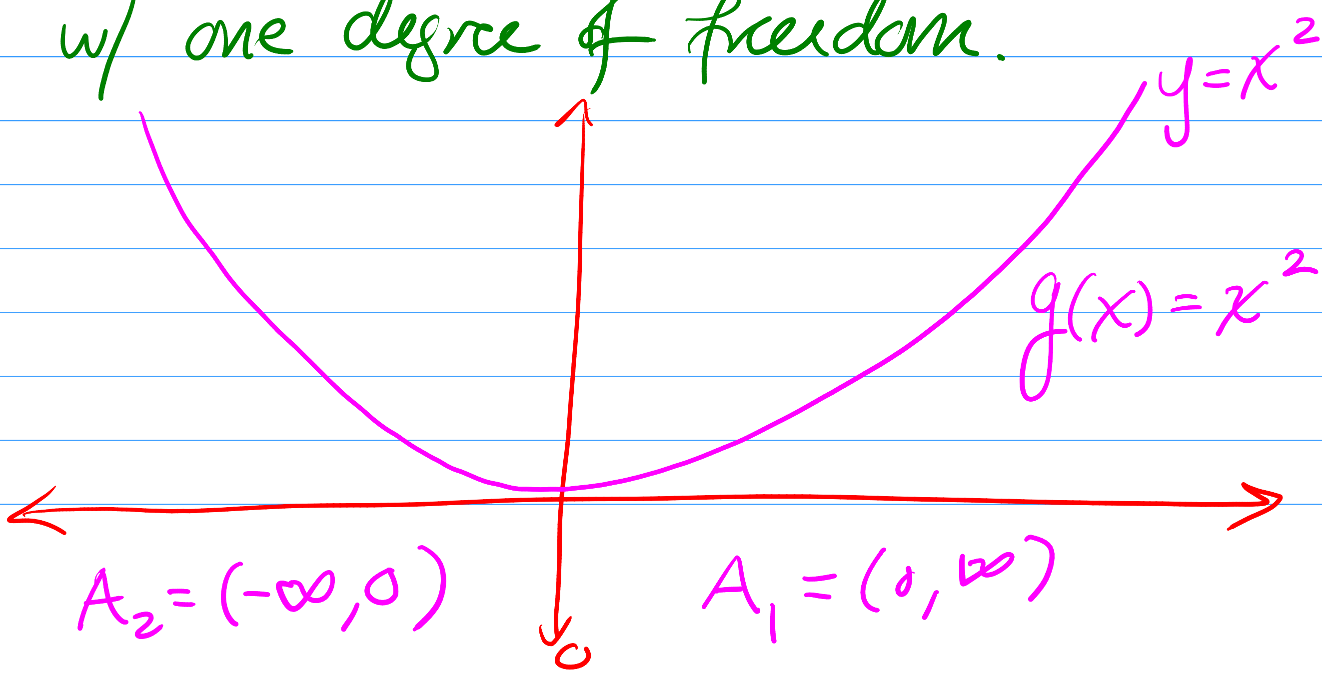
for $y \in \mathcal{Y}$

Ex: Chi-Squared Dist.

If $X \sim N(0,1)$ and $Y = X^2$

then Y has a Chi-Sq. dist.

w/ one degree of freedom.



$$A_1 = (0, \infty), \quad g_1(x) = x^2$$

$$g_1^{-1}(y) = \sqrt{y} \Rightarrow \frac{dg_1^{-1}}{dy} = \frac{1}{2\sqrt{y}}$$

$$A_2 = (-\infty, 0), \quad g_2(x) = x^2$$

$$g_2^{-1}(x) = -\sqrt{y} \Rightarrow \frac{dg_2^{-1}}{dy} = \frac{-1}{2\sqrt{y}}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

$$f_Y(y) = f_X(g_1^{-1}(y)) \left| \frac{dg_1^{-1}}{dy} \right| + f_X(g_2^{-1}(y)) \left| \frac{dg_2^{-1}}{dy} \right|$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\sqrt{y}^2\right) \left| \frac{1}{2\sqrt{y}} \right| + \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(-\sqrt{y})^2\right) \left| \frac{-1}{2\sqrt{y}} \right|$$

$$= \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y)\right) \frac{1}{2\sqrt{y}}, \quad y > 0$$

$$\boxed{= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} \exp\left(-\frac{1}{2}y\right), \quad y > 0}$$

Theorem: Prob. Integral Transf.

If X is cts w/ CDF F_X then

$$Y = F_X(X) \sim U(0,1).$$

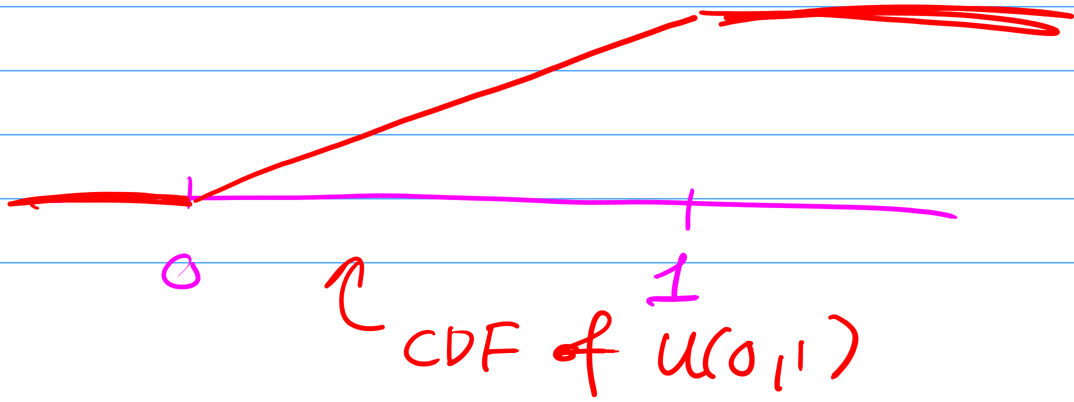
Pf. $\overset{=g}{F_X}$ is strictly increasing.
So $\underset{=g^{-1}}{F_X^{-1}}$ exists.

Our CDF theorem says if
$Y = g(X)$, g inc, then $F_Y(y) = F_X(g^{-1}(y))$

So

$$F_Y(y) = F_X(F_X^{-1}(y)) = y \leftarrow \text{CDF of } U(0,1)$$

for $0 < y < 1$



Know: how to generate $U(0,1)$

Want: generate RV w/ CDF F_X

Let $U \sim U(0,1)$

and $Z = F_X^{-1}(U)$

then $Z \sim F_X$.

This works:

$$F_Z(z) = P(Z \le z)$$
$$= P(F_X^{-1}(U) \le z)$$
$$= P(U \le F_X(z))$$
$$= F_U(F_X(z))$$
$$= F_X(z)$$

Ex. Want $X \sim Exp(1)$

CDF of $Exp(1)$ is $F_X(x) = 1 - e^{-x}, \; x > 0$

$$\boxed{F_X^{-1}(x) = -\log(1-x)}$$

Bivariate RVs

If $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$
then $Z = (X, Y)$ is called a
bivariate RV.

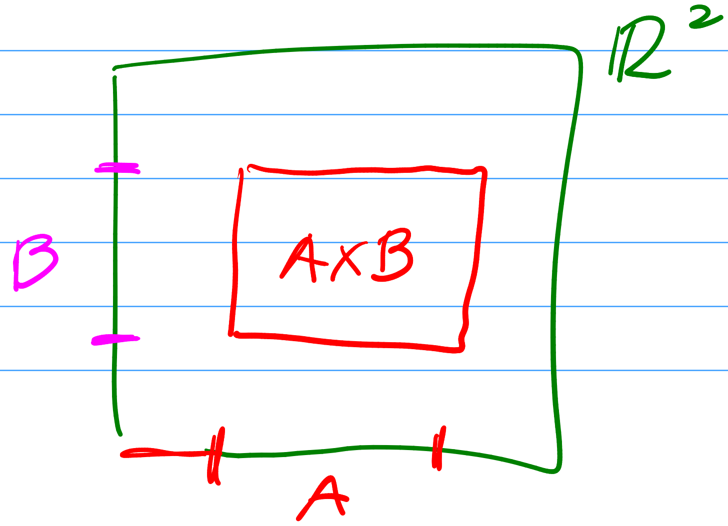
$Z : S \to \mathbb{R}^2$ s.t. $Z(s) = (X(s), Y(s))$.

$C \subset \mathbb{R}^2$

Say: $P(Z \in C)$
$= P((X, Y) \in C)$

Often, $C = A \times B$

$P((X,Y) \in C)$
$= P(X \in A, Y \in B)$



Ex. Flip a coin 3 times.

$$X = \begin{cases} 0, & \text{last flip } T \\ 1, & \text{last flip } H \end{cases}$$

Y = # heads among 3 flips

$Z = (X, Y)$

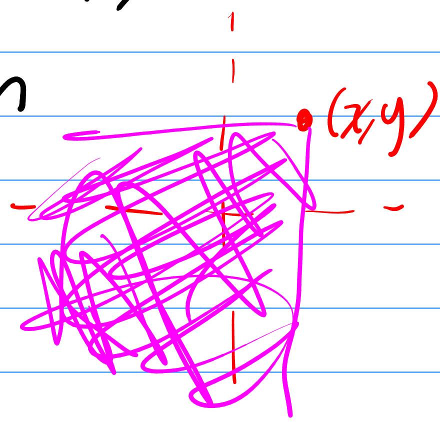
$s \in S$	$Z(s) \in \mathbb{R}^2$
H H H	(1, 3)
H H T	(0, 2)
H T H	(1, 2)
H T T	(0, 1)
T H H	(1, 2)
T H T	(0, 1)
T T H	(1, 1)
T T T	(0, 0)

Defn: Bivariate CDF (Joint CDF)

The joint CDF is a function

$$F : \mathbb{R}^2 \to \mathbb{R}$$

defined for (x,y) as



$$F(x,y) = P(X \le x, Y \le y)$$

Properties of Joint CDF

1. $F(x,y) \geq 0$

2. $\lim_{x,y \to \infty} F(x,y) = 1$

3. $\lim_{x \to -\infty} F(x,y) = 0$

   $\lim_{y \to -\infty} F(x,y) = 0$

4. F is non-decreasing and right-cts in each argument.