# Lecture 15

**Theorem:** If $X$ is a RV and $Y = g(X)$

where ① $g$ is invertible

② $g^{-1}$ is differentiable

then $$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dg^{-1}}{dy}\right|$$

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pf: **Case 1: $g$ increasing**

CDF theorem: $F_Y(y) = F_X(g^{-1}(y))$

$$f_Y(y) = \frac{dF_Y}{dy} = f_X(g^{-1}(y))\left|\frac{dg^{-1}}{dy}\right|$$

**Case 2: $g$ decreasing**

CDF theorem: $F_Y(y) = 1 - F_X(g^{-1}(y))$

$$f_Y(y) = \frac{dF_Y}{dy} = -f_X(g^{-1}(y))\,\underbrace{\frac{dg^{-1}}{dy}}_{<0}$$

$$= f_X(g^{-1}(y))\left|\frac{dg^{-1}}{dy}\right|$$

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**Ex.** $X \sim \text{Gamma}(k, \lambda)$

$$\boxed{f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)}, \quad x > 0}$$

let $Y = 1/X$

$$y = g(x) = 1/x \Rightarrow x = \boxed{\frac{1}{y} = g^{-1}(y)}$$

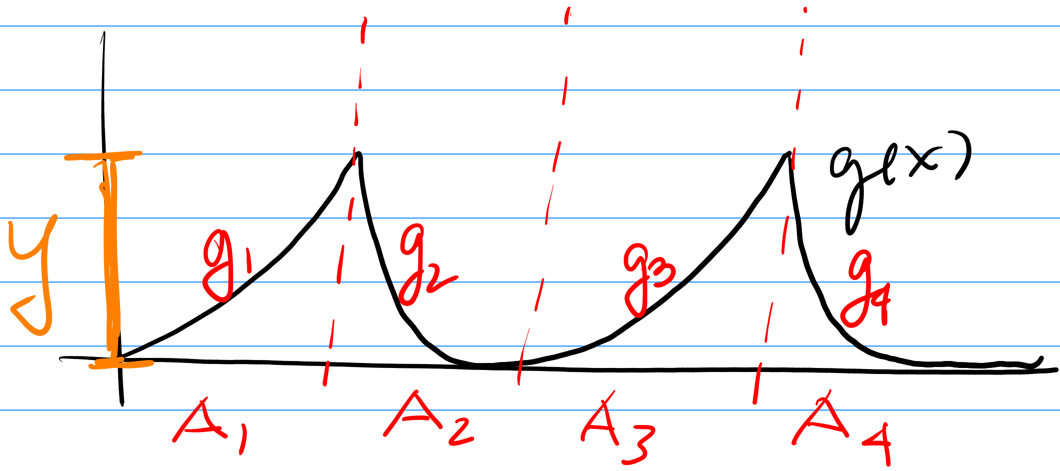
$$\frac{dg^{-1}}{dy} = -\frac{1}{y^2}$$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dg^{-1}}{dy}\right|$$

$$= f_X\left(\frac{1}{y}\right)\left|\frac{-1}{y^2}\right|$$

$$\boxed{f_Y(y) = \frac{\lambda e^{-\lambda/y}(\lambda/y)^{k-1}}{\Gamma(k)} \frac{1}{y^2}, \quad y>0}$$

$\longleftrightarrow$ Inverse Gamma dist




<u>Theorem:</u> Let $X$ be a cts RV w/ support $\mathcal{X}$ and let $A_1, \dots, A_K$ partition $\mathcal{X}$.

Let $g_i$ be $g$ restricted to $A_i$.

Let the range of $g_i$, called $\mathcal{Y}$, be the same for all $i$.

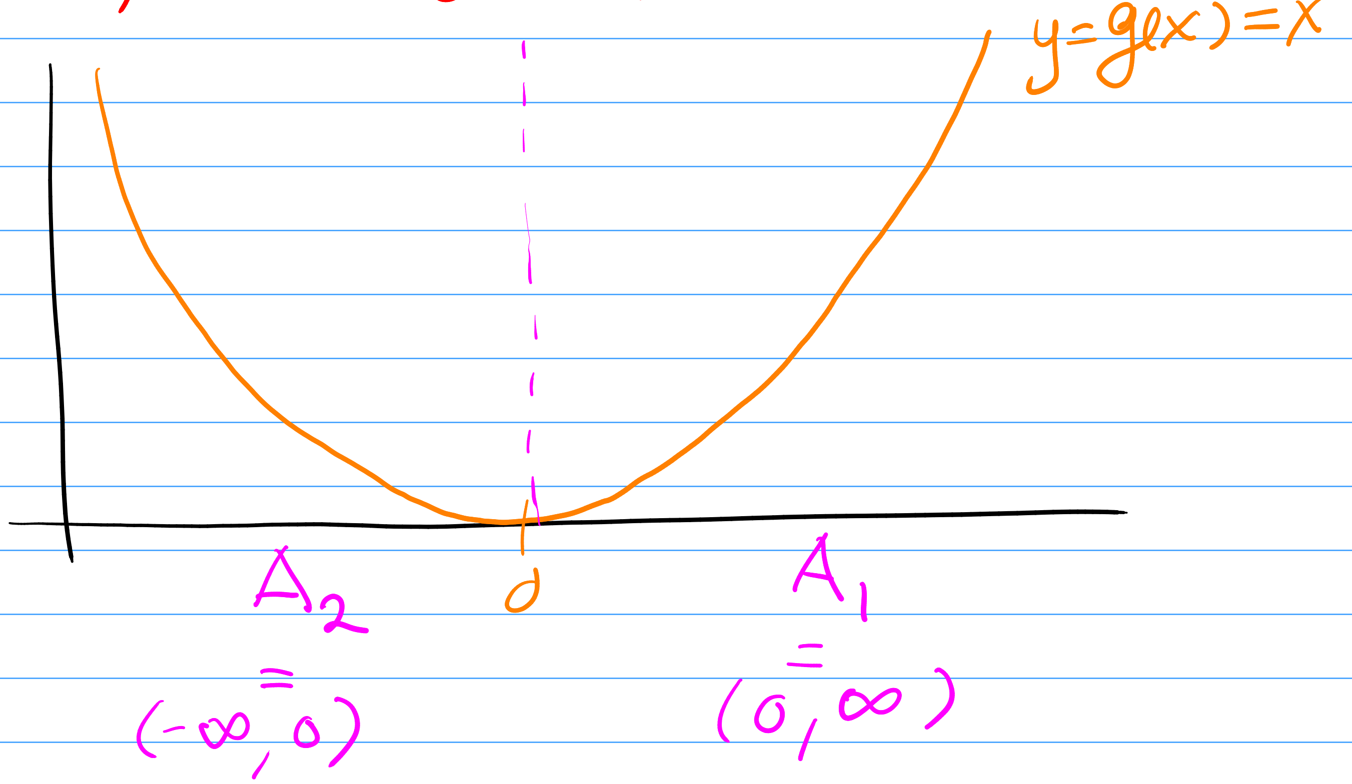
If my prev. theorem applies to each $g_i$, then

$$f_Y(y) = \sum_{i=1}^{K} f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}}{dy} \right|$$

**Ex. Chi-Squared dist**

If $X \sim N(0,1)$ and $Y = X^2$.

Then $Y$ has a chi-sq. dist, w/ one degree of freedom.



$$A_1 = (0, \infty), \quad g_1(x) = x^2$$

$$g_1^{-1}(y) = \sqrt{y}$$

$$\frac{dg_1^{-1}}{dy} = \frac{1}{2\sqrt{y}}$$

$$A_2 = (-\infty, 0), \quad g_2(x) = x^2$$

$$g_2^{-1}(y) = -\sqrt{y}$$

$$\frac{dg_2^{-1}}{dy} = -\frac{1}{2\sqrt{y}}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

$$f_Y(y) = f_X(g_1^{-1}(y)) \left|\frac{dg_1^{-1}}{dy}\right| + f_X(g_2^{-1}(y)) \left|\frac{dg_2^{-1}}{dy}\right|$$

$$= f_X(\sqrt{y}) \left|\frac{1}{2\sqrt{y}}\right| + f_X(-\sqrt{y}) \left|\frac{-1}{2\sqrt{y}}\right|$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(\sqrt{y})^2\right) \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(-\sqrt{y})^2\right) \frac{1}{2\sqrt{y}}$$

$$= \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y\right) \frac{1}{2\sqrt{y}}$$

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} \exp\left(-\frac{1}{2}y\right), \quad y > 0}$$

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**Theorem: Probability Integral Transf.**

If $X$ is cts w/ CDF $F_X$ then

$$F_X(X) \sim U(0,1).$$

pf. Assume that $F_X$ is strictly increasing. Then $F_X$ is invertible i.e. $F_X^{-1}$ exists.

Let $Y = F_X(X)$ i.e. $Y = g(X)$ when $g = F_X$.

Our CDF theorem says that

$$F_Y(y) = F_X(g^{-1}(y)) = F_X(F_X^{-1}(y))$$
$$= y, \quad 0 < y < 1$$

This is the CDF of a $U(0,1)$.

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**Know**: how to generate a $U(0,1)$ RV.

**Want**: generate nums from CDF $F_X$.

Let $\boxed{X = F_X^{-1}(U)}$ where $U \sim U(0,1)$

then $X \sim F_X$.

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**idea**: $F_X(X) \stackrel{d}{=} U \sim U(0,1)$

so $X \stackrel{d}{=} F_X^{-1}(U) \sim F_X$.

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**Ex**, Want $X \sim Exp(1)$

CDF of $Exp(1)$ is $F_X(x) = 1 - e^{-x}$, $x>0$

thus $F_X^{-1}(x) = -\log(1-x)$

# Bivariate RVs

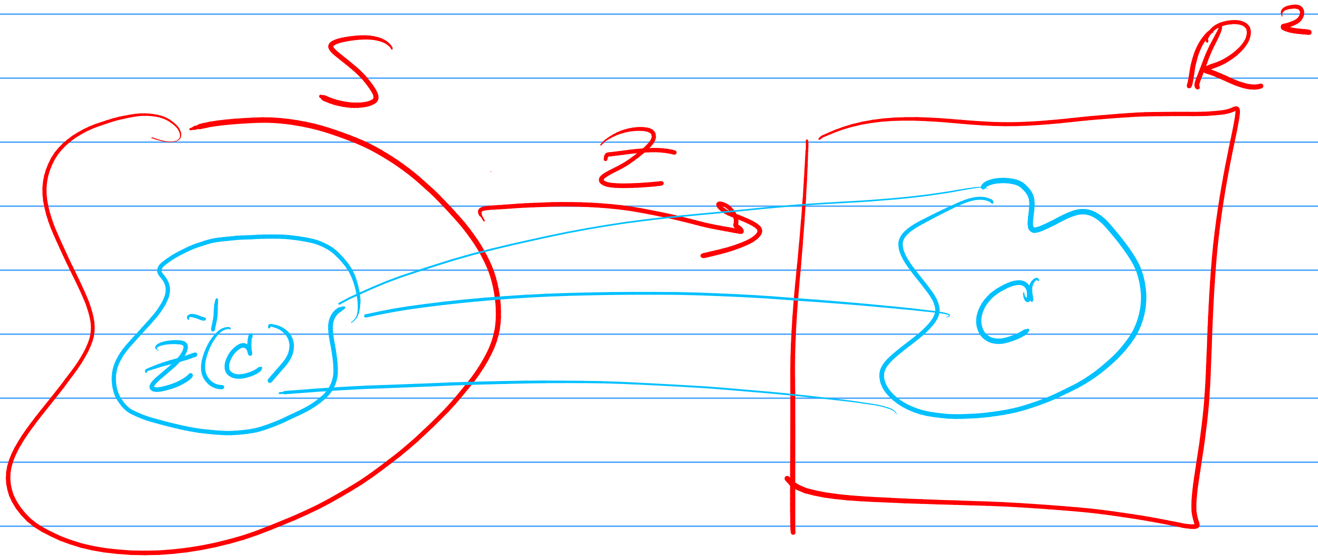
If $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$ then $Z = (X, Y)$ is called a bivariate RV.

So $Z : S \to \mathbb{R}^2$ where

$$Z(s) = (X(s), Y(s))$$

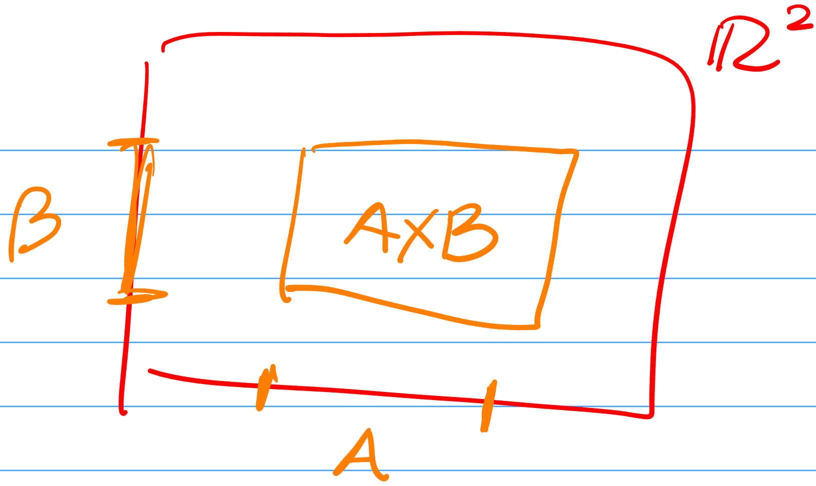
Say: $P(Z \in C)$ when $C \subset \mathbb{R}^2$

$$= P((X, Y) \in C)$$




Often, $C = A \times B$ when $A, B \subset \mathbb{R}$

$$P((X,Y)\in C)$$
$$= P(X\in A,\ Y\in B)$$

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Ex. Consider flipping coin 3 times.

$$X = \begin{cases} 0, & \text{last flip } T \\ 1, & \text{last flip } H \end{cases}$$

$Y =$ # heads among 3 flips

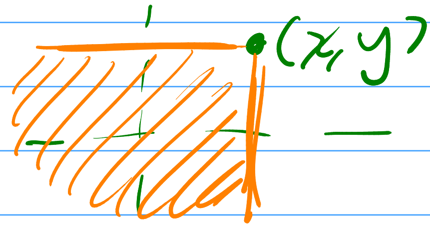
$Z = (X, Y)$

| $s \in S$ | $Z(s)$ |
|---|---|
| HHH | (1, 3) |
| HHT | (0, 2) |
| HTH | (1, 2) |
| HTT | (0, 1) |
| THH | (1, 2) |
| THT | (0, 1) |
| TTH | (1, 1) |
| TTT | (0, 0) |

Defn: Bivariate CDF (Joint CDF)

The joint CDF is a function

$$F: \mathbb{R}^2 \to \mathbb{R}$$




so that

$$F(x,y) = P(X \le x, Y \le y)$$

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Properties of Joint CDF

1. $F(x,y) \in [0,1]$

2. $\lim_{x,y \to \infty} F(x,y) = 1$

3. $\lim_{x \to -\infty} F(x,y) = 0$

   $\lim_{y \to -\infty} F(x,y) = 0$

4. F is non-decreasing and right-cts in each argument.

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