Lecture 16

Defn: Marginal Properties

If (X, Y) is bivariate RV then X and Y are called the marginal RVs and their corresp. props. are called marginal props.

Theorem: Rel. btwn Joint/Marginal CDFs

① $$F_X(x) = \lim_{y \to \infty} F(x,y)$$

② $$F_Y(y) = \lim_{x \to \infty} F(x,y)$$

Pf $$F_X(x) = P(X \leq x)$$
$$= P(X \leq x, Y = \text{anything})$$
$$= P(X \leq x, Y < \infty)$$

$$= \lim_{y\to\infty} P(X \le x, Y \le y)$$

$$= \lim_{y\to\infty} F(x,y)$$

Defn: Joint PMF

If X and Y are discrete then the joint PMF is defined as

$$f(x,y) = P(X=x, Y=y)$$

Theorem: Valid PMF

A fn f is a valid joint PMF iff

1. $f(x,y) \ge 0 \quad \forall x, y$
2. $\sum_x \sum_y f(x,y) = 1.$

Theorem: Rel. btwn Joint/Marginal PMF

① $f_X(x) = \sum_y f(x,y)$

② $f_Y(y) = \sum_x f(x,y)$

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pf: A_k partition S then $P(B) = \sum_k P(BA_k)$

Let $A_y = \{s : Y(s) = y\} \subset S$

Notice, A_y partition S

Let $B = $ "$X = x$"

then

$$f_X(x) = P(X=x) = P(B)$$

$$= \sum_y P(BA_y)$$

$$= \sum_y P(X=x, Y=y)$$

Ex. Flip 3 coins,

$$X = \begin{cases} 0 & \text{last } T \\ 1 & \text{last } H \end{cases}$$

$Y = \#$ heads

$X \backslash Y$	0	1	2	3	
0	$f(0,0) = 1/8$	$f(0,1) = 2/8$	$f(0,2) = 1/8$	$f(0,3) = 0$	$f_X(0) = 1/2$
1	0	$1/8$	$2/8$	$1/8$	$f_X(1) = 1/2$
	$f_Y(0) = \frac{1}{8}$	$f_Y(1) = 3/8$	$f_Y(2) = \frac{3}{8}$	$f_Y(3) = \frac{1}{8}$	

Defn: Joint PDF

If X and Y are cts then we call the function

$$f: \mathbb{R}^2 \to \mathbb{R}$$

the joint PDF if $\forall\, C \subset \mathbb{R}^2$

$$P((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$$

Facts!

1. $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$

2. $f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$

3. f is a valid Joint PDF iff

   i. $f(x,y) \geq 0 \quad \forall x,y$

   ii. $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$

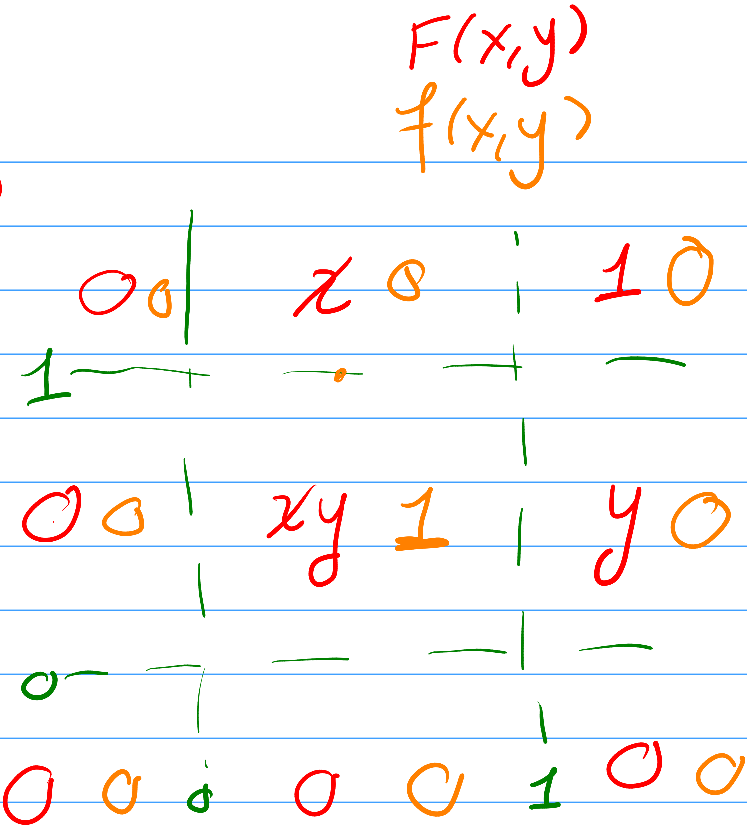
Theorem: Rel. btwn joint/marginal PDFs

1. $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$

2. $f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$

Ex,

$$F(x,y)=\begin{cases} 0, & x<0 \text{ or } y<0 \\ xy, & 0<x<1,\ 0<y<1 \\ x, & 0<x<1,\ y>1 \\ y, & 0<y<1,\ x>1 \\ 1, & x>1 \text{ and } y>1 \end{cases}$$



what's the joint PDF?

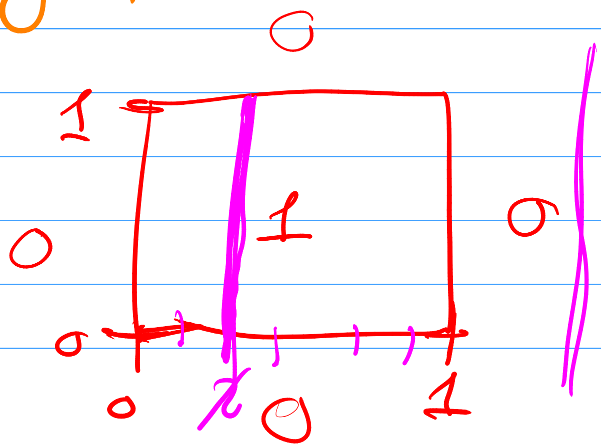
$$f(x,y)=\frac{\partial^2 F}{\partial x \partial y}$$

$$f(x,y)=1 \quad \text{for } 0<x<1,\ 0<y<1$$

what's the marginal density of X?

$$f_X(x)=\int_{\mathbb{R}} f(x,y)\,dy$$

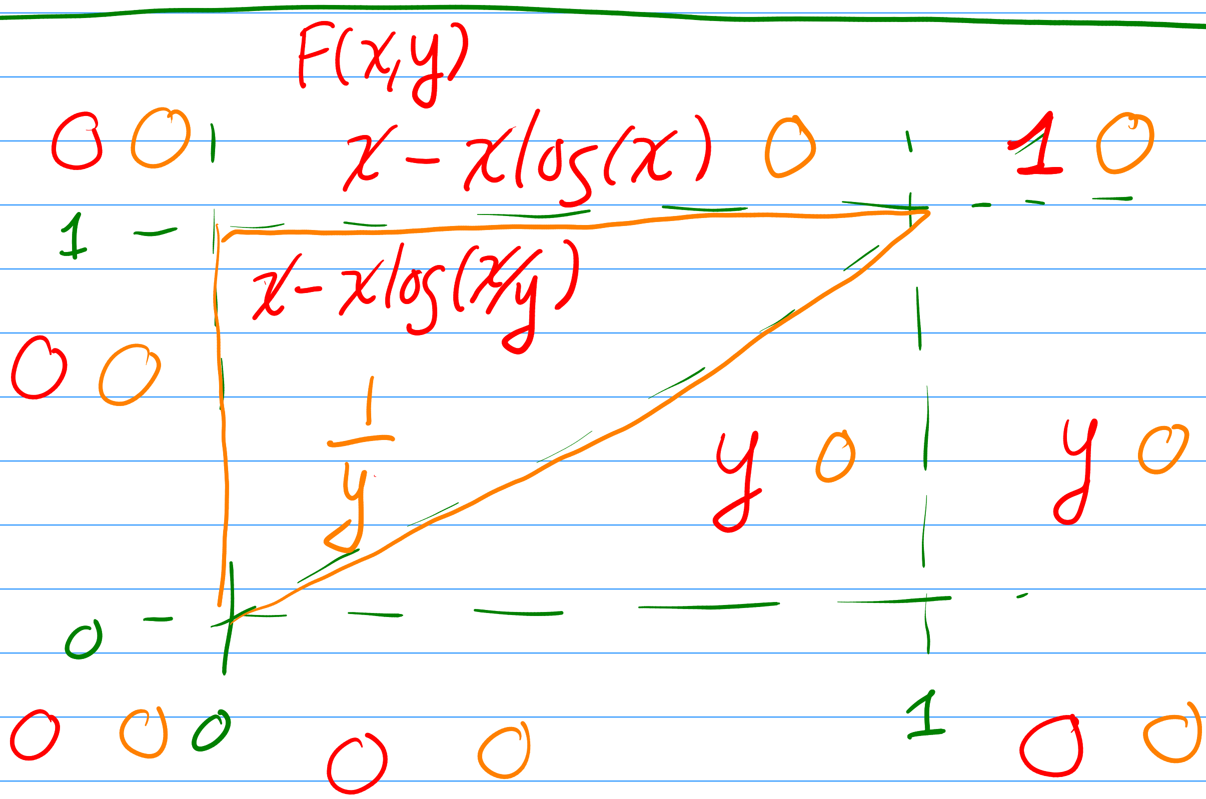
$$=\int_0^1 1\,dy$$



$= 1 \quad \text{for } 0 < x < 1$

$X \sim U(0,1)$

Ex.



Q: What's the joint PDF?

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left[x - x\log(x/y)\right]$$

$$= \frac{\partial}{\partial x}\left[-x\,\frac{-x/y^2}{x/y}\right]$$

$$= \frac{\partial}{\partial x}\left[x/y\right] = \frac{1}{y}$$

$$\boxed{\frac{\partial}{\partial x}\log(u(x)) = \frac{u'(x)}{u(x)}}$$

$$f(x,y) = \frac{1}{y} \quad \text{for} \quad 0 < x < y < 1$$

Q: What's marginal of X?

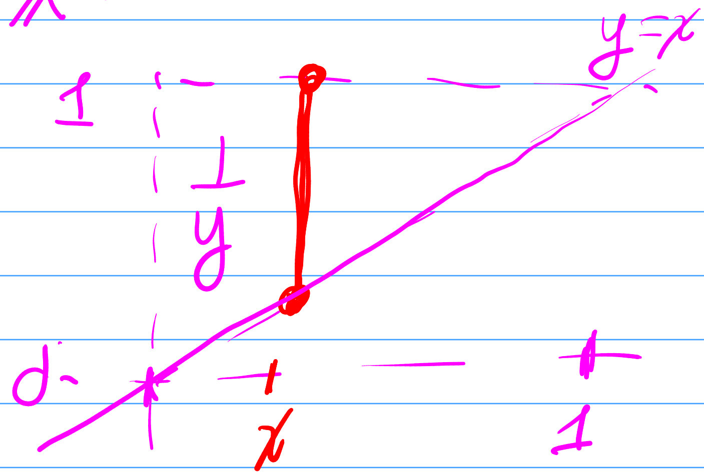
For $0 < x < 1$

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$$

$$= \int_x^1 \frac{1}{y}\,dy$$

$$= \log(y) \Big|_x^1 = 0 - \log(x)$$

$$\boxed{= -\log(x) \quad \text{for} \quad 0 < x < 1.}$$

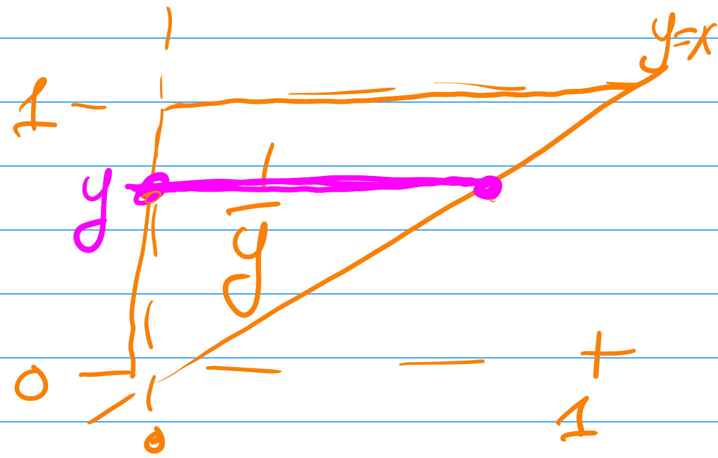


Q: What's marginal of Y?

For $0 < y < 1$

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$$

$$= \int_0^y \frac{1}{y}\,dx$$



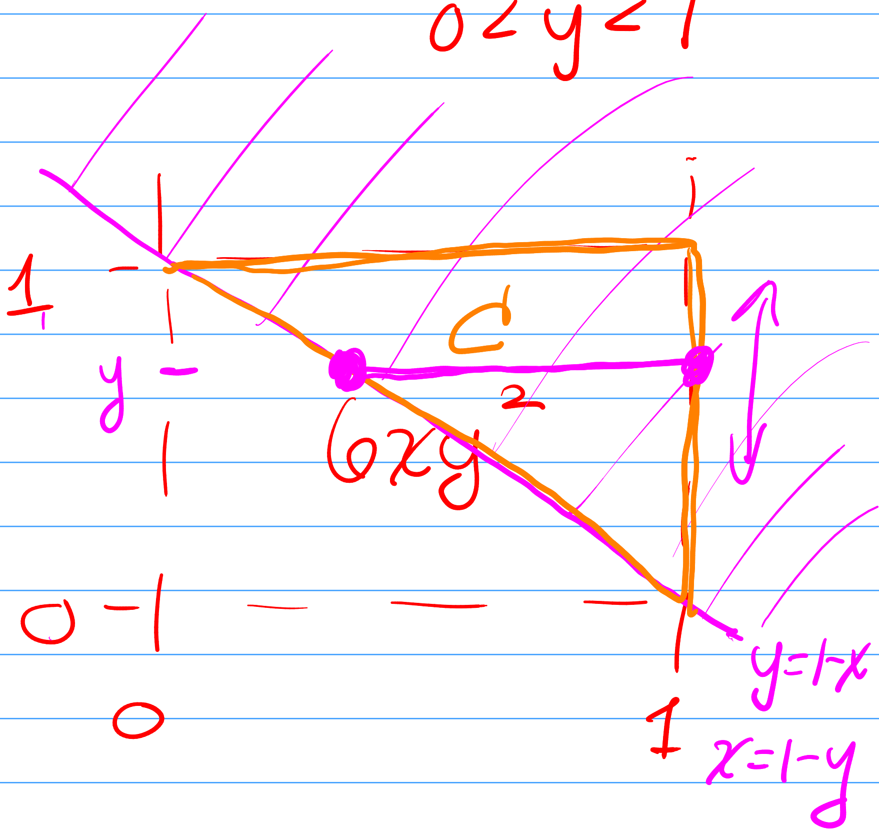
$$= \frac{x}{y}\Big|_0^y = 1 - 0 = 1$$

$$\boxed{f_Y(y) = 1 \quad \text{for } 0 < y < 1.}$$

Ex. Let $f(x,y) = 6xy^2$ for $0 < x < 1$, $0 < y < 1$

Let's find $P(X+Y \geq 1)$

draw line $x+y=1$



$$P(X+Y \geq 1)$$

$$= \iint_C f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_{1-y}^1 6xy^2\,dx\,dy = \int_0^1 6y^2 \left[\frac{x^2}{2}\right]_{1-y}^1 dy$$

$$= \int_0^1 3y^2 \left(1 - (1-y)^2\right) dy$$

$$= \cdots$$

$$= 9/10$$