Lecture 16

Defn: Marginal Properties

If (X, Y) is a biv. RV. then X and Y individually are called the marginal RVs.

Their properties are called marginal properties.

Theorem: Rel. btwn Joint/Marginal CDFs

1. $F_X(x) = \lim_{y\to\infty} F(x,y)$

2. $F_Y(y) = \lim_{x\to\infty} F(x,y)$

$$F_X(x) = P(X \le x)$$

$$= P(X \le x, Y < \infty)$$

$$= \lim_{y\to\infty} P(X \le x, Y \le y)$$

$$= \lim_{y\to\infty} F(x,y)$$

Defn: Joint PMF

If X and Y are discrete then the joint PMF is

$$f(x,y) = P(X=x, Y=y)$$

[uni: $f(x) = P(X=x)$]

Theorem: Valid PMF

A fn f is a valid PMF iff

1. $f(x,y) \geq 0$
2. $\sum_x \sum_y f(x,y) = 1$.

Theorem: Rel. btwn Joint/Marginal PMFs

1. $f_X(x) = \sum_y f(x,y)$

② $f_Y(y) = \sum_x f(x,y)$

Pf. Recall: A_k partition S, $P(B) = \sum_k P(BA_k)$

let $A_y = \{s : Y(s) = y\} \subset S \quad \forall y$

The A_y partition S.

$$f_X(x) = P(\underbrace{X=x}_{B}) = \sum_y P(BA_y)$$

$$= \sum_y P(X=x, Y=y)$$

$$= \sum_y f(x,y)$$

Ex. Flip 3 coins,

$$X = \begin{cases} 0, & \text{last } T \\ 1, & \text{last } H \end{cases}$$

$Y = \#\text{heads}$

$f(x,y)$

$X \backslash Y$	0	1	2	3	
0	$f(0,0)=1/8$	$f(0,1)=2/8$	$f(0,2)=1/8$	$f(0,3)=0$	$f_X(0)=1/2$
1	0	$1/8$	$2/8$	$1/8$	$f_X(1)=1/2$
	$f_Y(0)=\frac{1}{8}$	$f_Y(1)=\frac{3}{8}$	$f_Y(2)=\frac{3}{8}$	$f_Y(3)=\frac{1}{8}$	

Defn: Joint PDF

If X and Y are cts we call the function $f: \mathbb{R}^2 \to \mathbb{R}$ the joint PDF if $\forall C \subset \mathbb{R}^2$

$$P((X,Y)\in C) = \iint_C f(x,y)\,dx\,dy.$$

[Uni: $P(X\in A) = \int_A f(x)\,dx$]

Facts:

1. $$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$$

$$\left[\text{Uni}: F(x) = \int_{-\infty}^{x} f(t)\,dt\right]$$

2. $$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

$$\left[\text{Uni}: f(x) = \frac{\partial F}{\partial x}\right]$$

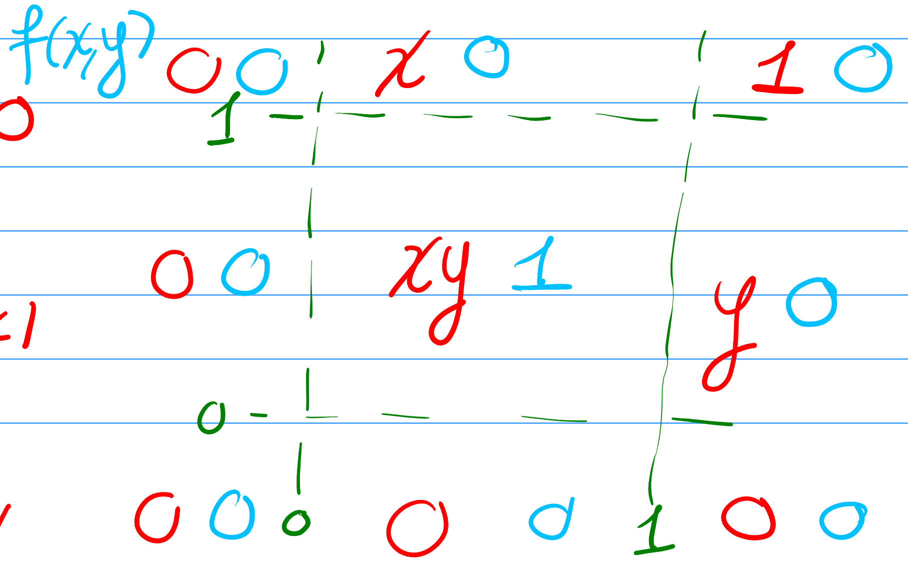
3. f is a valid joint PDF iff

   (i) $f(x,y) \geq 0 \quad \forall x,y$

   (ii) $\iint_{R^2} f(x,y)\,dx\,dy = 1$

Ex.

$$F(x,y) = \begin{cases} 0, & x<0 \text{ or } y<0 \\ 1, & x>1 \text{ and } y>1 \\ xy, & 0<x<1,\ 0<y<1 \\ x, & 0<x<1,\ y>1 \\ y, & 0<y<1,\ x>1 \end{cases}$$



Q: Whats the joint PDF?

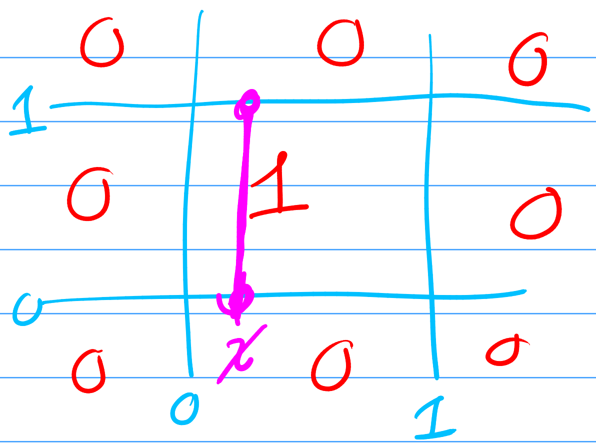
$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

$$f(x,y) = 1 \text{ for } 0<x<1,\ 0<y<1$$

Can say (X,Y) is uniform over the unit-square.

Q: What's the marginal density of X?

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$$



for $0<x<1$

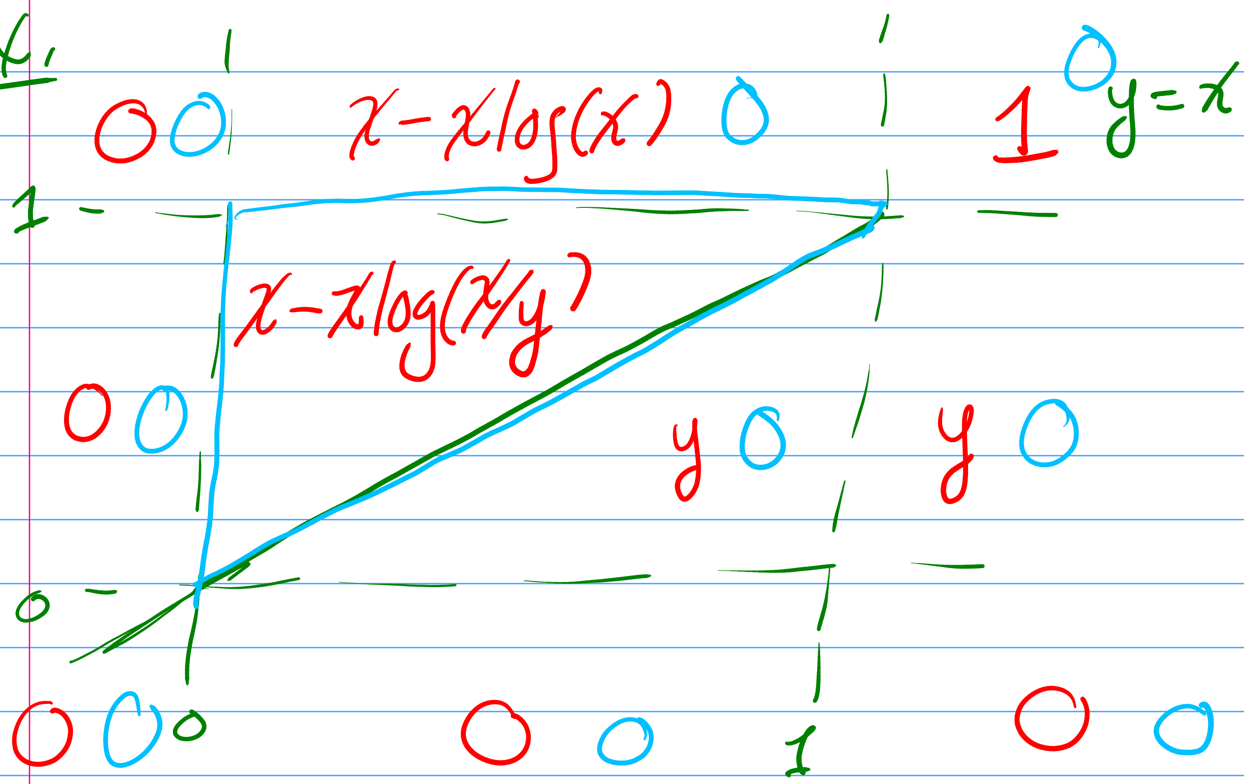
$$= \int_0^1 1\,dy$$

$$= 1$$

i.e. $X \sim U(0,1)$.

$F(x,y)$ $f(x,y)$

Ex.



Q: what's the joint PDF?

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

$$\frac{\partial}{\partial x}\log(u(x)) = \frac{u'(x)}{u(x)}$$

For $0 < x < y < 1$

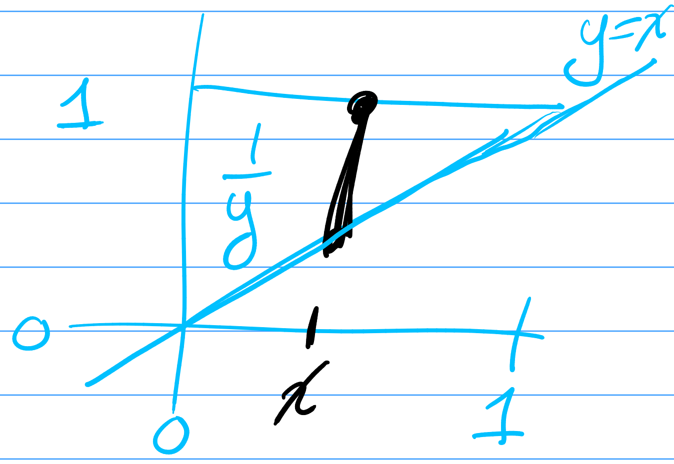
$$f(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left[x - x\log(x/y)\right]$$

$$= \frac{\partial}{\partial x}\left[-x\,\frac{-x/y^2}{x/y}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{x}{y}\right] = \frac{1}{y}$$

Q: what's the marginal of X?

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\, dy$$



For $0<x<1$

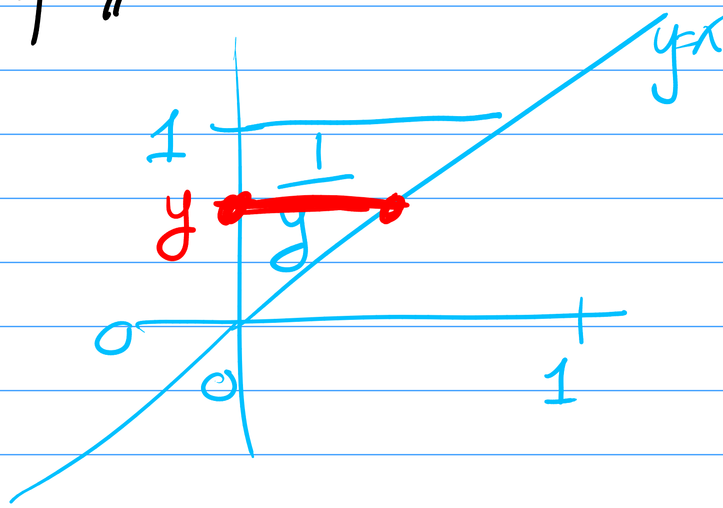
$$= \int_x^1 \frac{1}{y}\, dy$$

$$= \log(y)\Big|_x^1 = 0 - \log(x)$$

$$\boxed{f_X(x) = -\log(x) \text{ for } 0<x<1}$$

Q: what's the marginal of Y?

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\, dx$$



For $0<y<1$

$$= \int_0^y \frac{1}{y}\, dx$$

$$= \frac{1}{y}\, x \Big|_0^y = \frac{y}{y} - \frac{0}{y} = 1$$

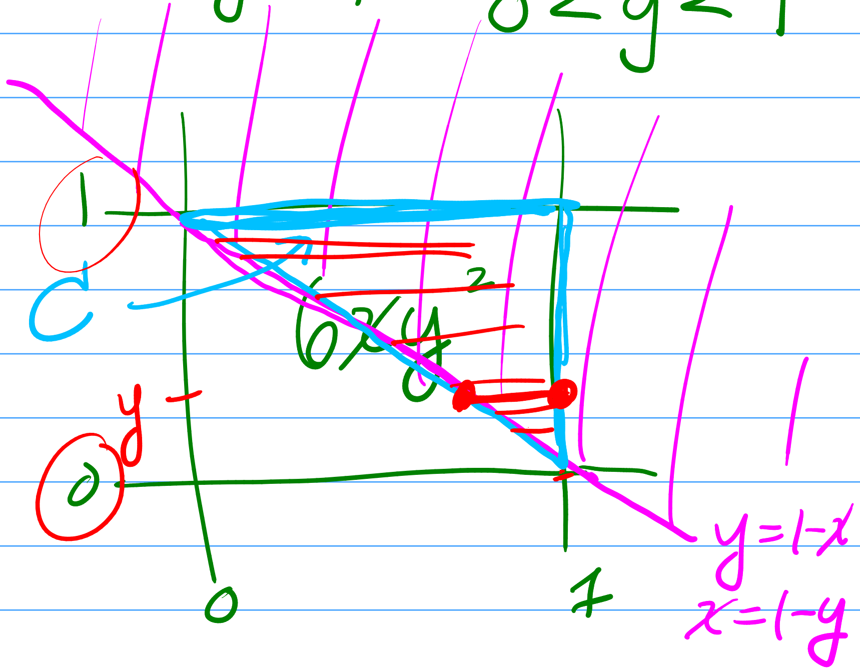
So $\boxed{f_Y(y) = 1 \quad \text{for } 0 < y < 1}$

$Y \sim U(0,1)$.

__Ex.__ Let $f(x,y) = 6xy^2$ for $0 < x < 1$, $0 < y < 1$

What is $P(X+Y \geq 1)$?

draw line $\boxed{x+y=1}$



$$= \iint_C f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_{x=1-y}^{x=1} 6xy^2\,dx\,dy$$

$$= \int_0^1 6y^2 \left[\frac{x^2}{2}\right]_{1-y}^{1} dy$$

$$= \int_0^1 3y^2\left(1-(1-y)^2\right) dy = \dots = 9/10$$