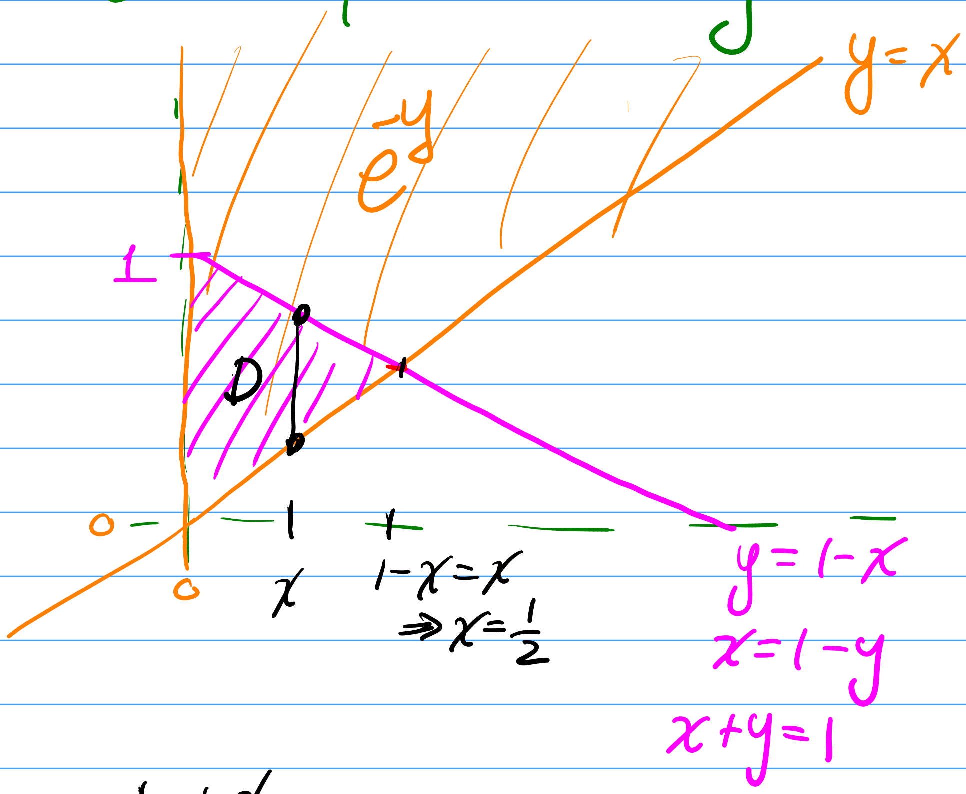
Lecture 17

Ex. $f(x,y) = e^{-y}$ for $0 < x < y$



$$P(X+Y \geq 1) = \iint_C f(x,y)\,dx\,dy$$

$$= 1 - P(X+Y<1)$$

$$= \iint_D f(x,y)\,dx\,dy = \int_0^{\frac{1}{2}}\int_x^{1-x} e^{-y}\,dy\,dx$$

$$= \int_0^{\frac{1}{2}} \Big[-e^{-y}\Big]_x^{1-x} dx$$

$$= \int_0^{\frac{1}{2}} -e^{-(1-x)} + e^{-x}\,dx$$

$$= \int_0^{\frac{1}{2}} e^{-x} - \frac{1}{e}e^{x}\,dx$$

$$= \left[-e^{-x} - \frac{1}{e} e^{x} \right]_0^{\frac{1}{2}}$$

$$\boxed{= -e^{-\frac{1}{2}} - \frac{1}{e} e^{1/2} - \left(-1 - \frac{1}{e} \right)}$$

Final answ:

$$P(X + Y \geq 1) = 1 -$$

Defn: Bivariate Expectation

If (X, Y) is a bivariate RV and

$$g : \mathbb{R}^2 \to \mathbb{R}$$

then

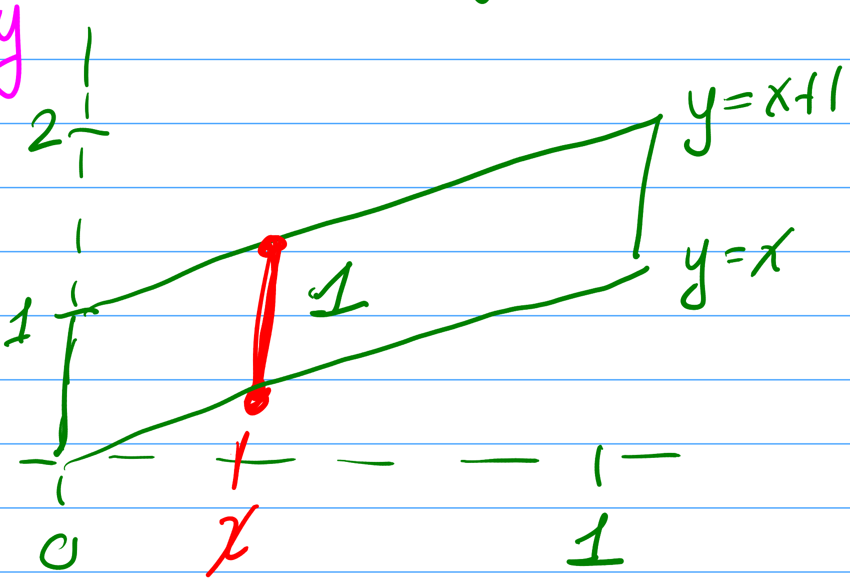
$$E[g(X,Y)] = \begin{cases} \sum_x \sum_y g(x,y) f(x,y) & \text{(discrete)} \\ \iint g(x,y) f(x,y)\, dx\, dy & \text{(cts)} \end{cases}$$

[Uni: $E[g(X)] = \int g(x) f(x)\, dx$]

Ex. Let $f(x,y) = 1$ over $0 < x < 1$

$x < y < x+1$

$$E[XY] = \iint xy\, f(x,y)\, dx\, dy$$



$$= \int_0^1 \int_x^{x+1} xy(1)\, dy\, dx$$

$$= \int_0^1 x \int_x^{x+1} y\, dy\, dx$$

$$= \int_0^1 x \left[\frac{y^2}{2}\right]_x^{x+1} dx = \int_0^1 \frac{x}{2}\left((x+1)^2 - x^2\right) dx$$

$$= \cdots = 7/12$$

Theorem: Biv. Exp. is Linear.

If $g_1 : \mathbb{R}^2 \to \mathbb{R}$, $g_2 : \mathbb{R}^2 \to \mathbb{R}$ and $a, b \in \mathbb{R}$

then

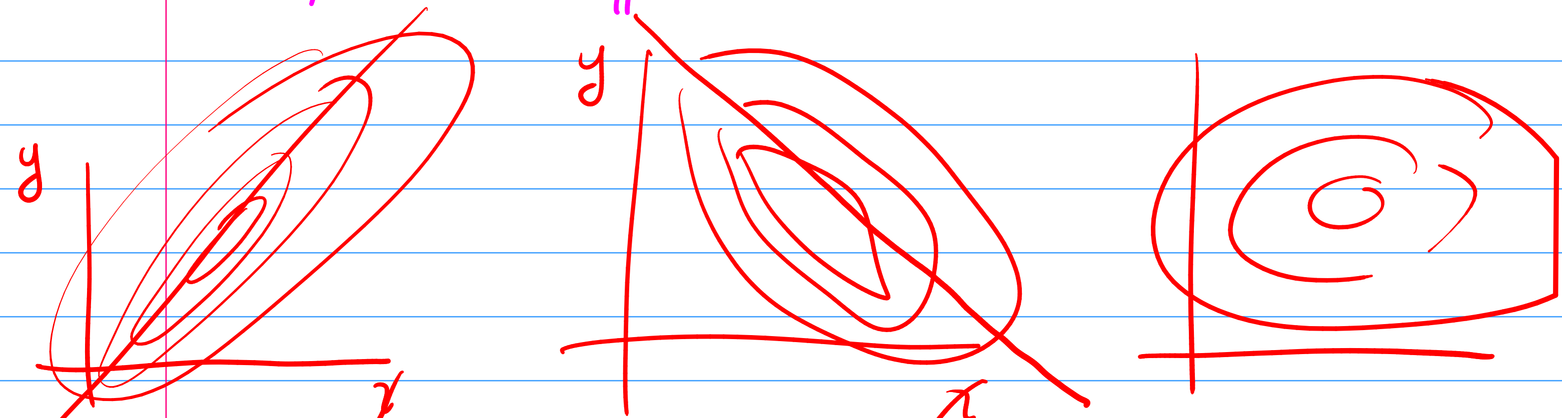
$$E[a g_1(X,Y) + b g_2(X,Y)] = a E[g_1(X,Y)] + b E[g_2(X,Y)].$$

Defn: Covariance

The covariance between X and Y is

$$Cov(X,Y) = E\left[(X - EX)(Y - EY)\right]$$
$$= E\left[(X - \mu_X)(Y - \mu_Y)\right]$$

Claim: Cov measures how linearly related X and Y are.



Note: ① $Var(X) = E\left[(X - EX)^2\right]$
$= Cov(X,X)$

② Covariance is scale sensitive:

$$Cov(5X, Y) = 5\, Cov(X,Y)$$

Defn: Correlation

Basically cov re-scaled to btwn
-1 and 1.

$$\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$$

$$= \frac{\text{Cov}(X, Y)}{\text{sd}(X)\,\text{sd}(Y)}.$$

<u>idea:</u> Cor ≈ 1, strong lin. rel.
Cor ≈ -1, strong (neg.) lin. rel.
Cor ≈ 0, no lin. rel.

<u>Theorem:</u> $a, b \in \mathbb{R}$,

$$\text{Var}(aX + bY) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y) + 2ab\,\text{Cov}(X, Y).$$

Pf: $Z = aX + bY$

$$Var(Z) = E[(Z - EZ)^2]$$

$$= E[(aX + bY - \underbrace{E[aX + bY]})^2]$$

$$= E[(aX + bY - aE[X] - bE[Y])^2]$$

$$= E[(\underbrace{a(X - EX)}_{\alpha} + \underbrace{b(Y - EY)}_{\beta})^2]$$

$$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

$$= E[a^2(X - EX)^2 + b^2(Y - EY)^2 + 2a(X - EX)b(Y - EY)]$$

$$= a^2 \underbrace{E[(X - EX)^2]}_{Var(X)} + b^2 \underbrace{E[(Y - EY)^2]}_{Var(Y)} + 2ab \underbrace{E[(X - EX)(Y - EY)]}_{Cov(X,Y).}$$

Theorem: $\mathrm{Cov}(aX+b, Y) = a\,\mathrm{Cov}(X,Y)$

Recall: $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$

pf. $\mathrm{Cov}(aX+b, Y)$

$$= E[(aX+b - E[aX+b])(Y-EY)]$$

$$= E[(aX+\cancel{b} - aEX - \cancel{b})(Y-EY)]$$

$$= E[a(X-EX)(Y-EY)]$$

$$= a\,E[(X-EX)(Y-EY)]$$

$$= a\,\mathrm{Cov}(X,Y).$$

Corollaries:

1. $\mathrm{Cov}(X, cY+d) = c\,\mathrm{Cov}(X,Y)$
2. $\boxed{\mathrm{Cov}(aX+b, cY+d) = ac\,\mathrm{Cov}(X,Y).}$

Theorem:

$$\mathrm{Cor}(aX+b, cY+d) = \mathrm{sgn}(a)\,\mathrm{sgn}(c)\,\mathrm{Cor}(X,Y)$$

$$\mathrm{sgn}(x) = \begin{cases} +1, & x>0 \\ -1, & x<0 \\ 0, & x=0 \end{cases}$$

Ex. $\mathrm{Cor}(-5X, Y) = -\mathrm{Cor}(X,Y)$

Pf. $a, c \neq 0$

note: $x \neq 0$ then $\mathrm{sgn}(x) = \frac{x}{|x|}$

$$\mathrm{Cor}(aX+b, cY+d) = \frac{\mathrm{Cov}(aX+b, cY+d)}{\sqrt{\mathrm{Var}(aX+b)\,\mathrm{Var}(cY+d)}}$$

$$= \frac{ac\,\mathrm{Cov}(X,Y)}{\sqrt{a^2\,\mathrm{Var}(X)\,c^2\,\mathrm{Var}(Y)}}$$

$$= \frac{a}{\sqrt{a^2}}\,\frac{c}{\sqrt{c^2}}\,\frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

$$\frac{a}{|a|} = \mathrm{sgn}(a) \qquad \mathrm{sgn}(c) \qquad \mathrm{Cor}(X, Y)$$

Theorem: $-1 \leq \mathrm{Cor}(X, Y) \leq 1$

WLOG: assume $\mathrm{Var}(X) = \mathrm{Var}(Y) = 1$

$E[X] = E[Y] = 0$

Note: $\mathrm{Cor}(X, Y) = \dfrac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \mathrm{Cov}(X, Y)$

Consider

$$\mathrm{Var}(X \pm Y) = \overbrace{\mathrm{Var}(X)}^{1} + \overbrace{\mathrm{Var}(Y)}^{1} \pm 2\mathrm{Cov}(X, Y)$$

≥ 0

So $2 \pm 2\mathrm{Cov}(X, Y) \geq 0$

$\Rightarrow 1 \pm \mathrm{Cor}(X, Y) \geq 0$

$1 + \mathrm{Cor} \geq 0$

$\boxed{\mathrm{Cor} \geq -1}$

$1 - \mathrm{Cor} \geq 0$

$\boxed{\mathrm{Cor} \leq 1}$

Theorem: Short-cut Covariance

$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y].$$

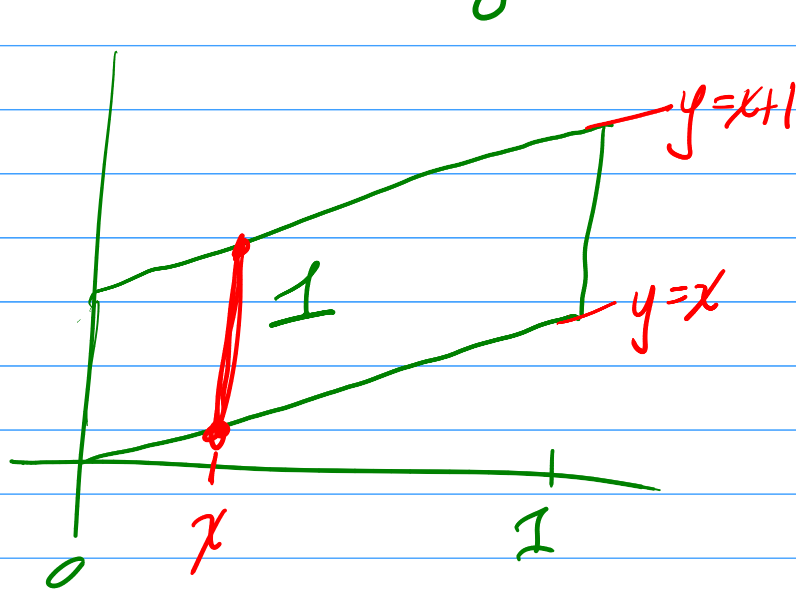
$\left(\text{Var:} \quad \mathrm{Var}(X) = E[X^2] - E[X]^2 \right)$

Ex. Continue: $f(x,y) = 1$ $\quad 0 < x < 1$, $\quad x < y < x+1$

Had calc:

$E[XY] = 7/12$

What's the Cov / Cor.



Marginal of X

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_x^{x+1} 1\,dy = y\Big|_x^{x+1}$$

$$= x+1-x = 1$$

So $f_X(x) = 1$ for $0 < x < 1$

$X \sim U(0,1)$

$E[X] = \frac{1}{2}$, $Var(X) = 1/12$.