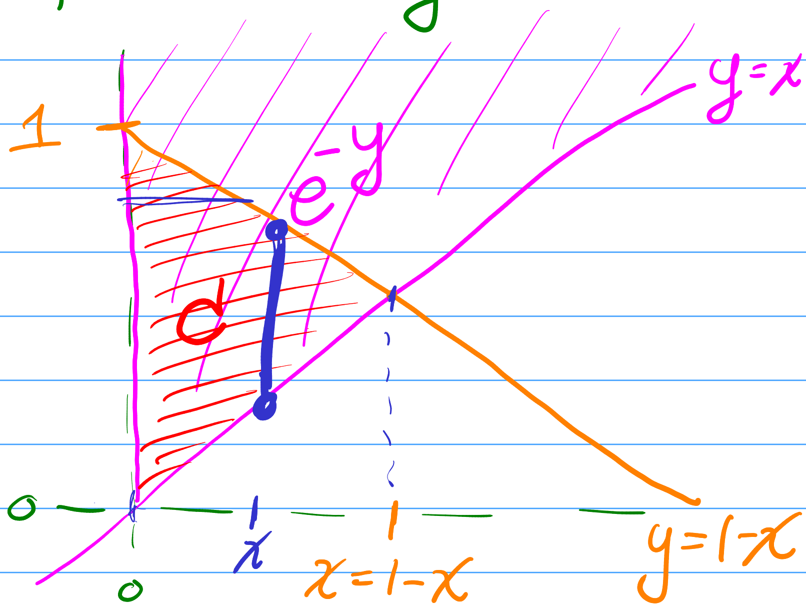
Lecture 17

Ex. $f(x,y) = e^{-y}$ for $0 < x < y$



$P(X + Y \leq 1)$

$x + y = 1 \Rightarrow y = 1 - x$

$x = 1 - x \Rightarrow x = \frac{1}{2}$

$$= \iint_d f(x,y)\,dx\,dy$$

$$= \int_0^{1/2}\int_x^{1-x} e^{-y}\,dy\,dx$$

$$= \int_0^{1/2} \left[-e^{-y}\right]_x^{1-x} dx$$

$$= \int_0^{\frac{1}{2}} -e^{-(1-x)} + e^{-x}\,dx$$

$$= \int_0^{\frac{1}{2}} -\frac{1}{e}e^{x} + e^{-x}\,dx = \left[-\frac{1}{e}e^{x} - e^{-x}\right]_0^{\frac{1}{2}}$$

$$\boxed{= -\frac{1}{e}e^{1/2} - e^{-1/2} + \frac{1}{e} + 1}$$

Defn: Bivariate Expectation

If (X, Y) is a Biv. RV and

$$g: \mathbb{R}^2 \to \mathbb{R}$$

then

$$E[g(X,Y)] = \begin{cases} \sum_x \sum_y g(x,y) f(x,y) & \text{(discrete)} \\ \iint g(x,y) f(x,y)\,dx\,dy & \text{(cts)} \end{cases}$$

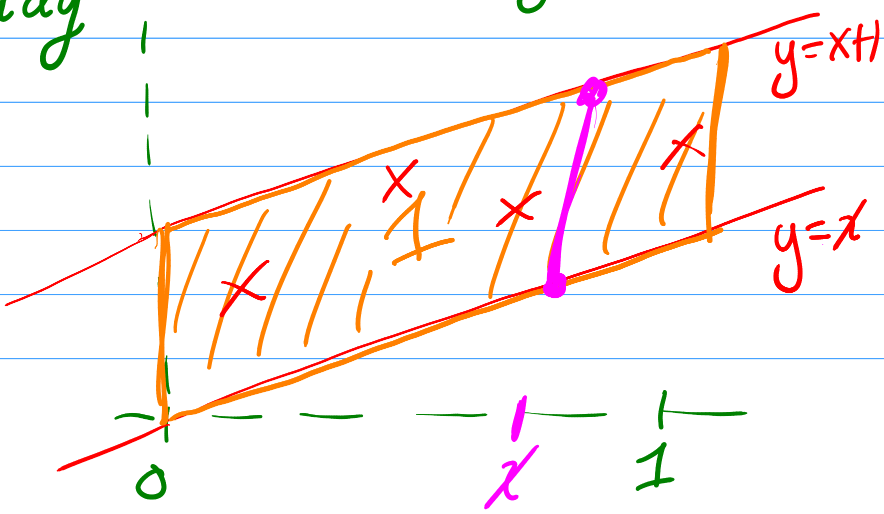
$$\left[\text{Uni}: \; E[g(X)] = \int g(x) f(x)\,dx \right]$$

Ex. Let $f(x,y) = 1$ over $0 < x < 1$

$x < y < x+1$

$$E[XY] = \iint xy\, f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_x^{x+1} xy\,(1)\,dy\,dx$$



$$= \int_0^1 x\left[\frac{y^2}{2}\right]_x^{x+1} dx$$

$$= \int_0^1 \frac{x}{2}\left[(x+1)^2 - x^2\right] dx$$

$$= \cdots = 7/12$$

Theorem: Biv. Exp. is Linear

If $g_1 : \mathbb{R}^2 \to \mathbb{R}$ and $g_2 : \mathbb{R}^2 \to \mathbb{R}$ and $a, b \in \mathbb{R}$ then

$$E\left[a g_1(X,Y) + b g_2(X,Y)\right] = aE[g_1(X,Y)] + bE[g_2(X,Y)].$$

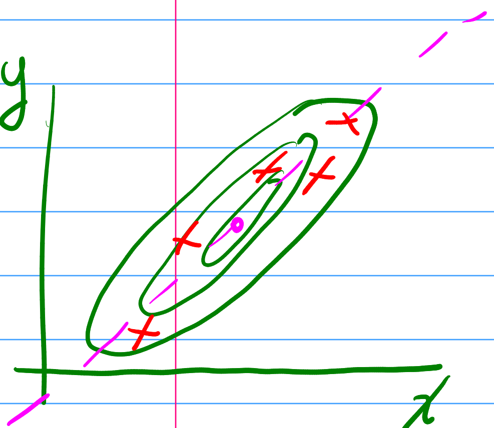
Defn: Covariance

The covariance between X and Y is

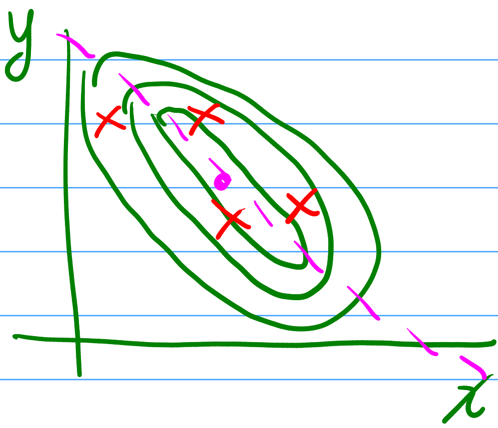
$$\mathrm{Cov}(X,Y) = E\left[(X - EX)(Y - EY)\right]$$

$$= E\left[(X - \mu_X)(Y - \mu_Y)\right]$$

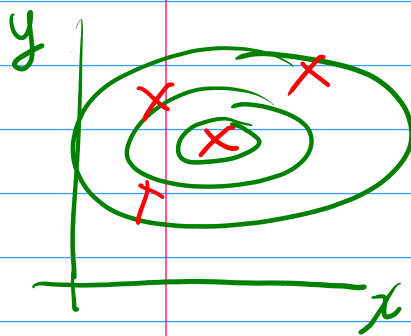
Claim: Cov measures strength/direction of linear rels. (btwn X and Y)



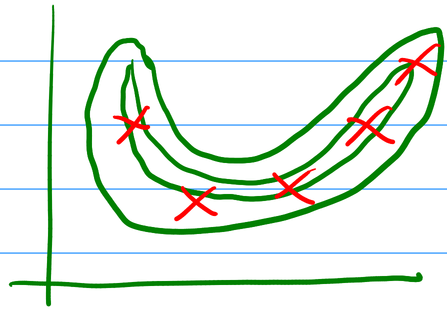
$\mathrm{cov} \gg 0$



$\mathrm{cov} \ll 0$



$\mathrm{cov} \approx 0$



$\mathrm{Cov} \approx 0$

(X, Y related, non-linearly)

Notes:

① $\mathrm{Var}(X) = E[(X - EX)^2]$
$= \mathrm{Cov}(X, X)$

② Covariance is scale sensitive:

$$\text{Cov}(5X, Y) = 5\,\text{Cov}(X, Y).$$

Defn: Correlation

Basically, re-scaled cov. to be btwn -1 and 1.

$$\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$$

$$= \frac{\text{Cov}(X, Y)}{sd(X)\,sd(Y)}.$$

idea: cor ≈ 1, strong lin rel

cor ≈ -1, strong neg. lin. rel.

cor ≈ 0, no strong lin. rel.

Theorem: If $a, b \in \mathbb{R}$

$$\operatorname{Var}(aX+bY) = a^2\operatorname{Var}(X) + b^2\operatorname{Var}(Y) + 2ab\operatorname{Cov}(X,Y)$$

pf. $Z = aX + bY$

$$\operatorname{Var}(Z) = E[(Z - EZ)^2]$$

$$= E[(aX+bY - E[aX+bY])^2]$$

$$= E[(aX+bY - aEX - bEY)^2]$$

$$= E[(\underbrace{a(X-EX)}_{\alpha} + \underbrace{b(Y-EY)}_{\beta})^2]$$

$$(\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

$$= E[a^2(X-EX)^2 + b^2(Y-EY)^2 + 2a(X-EX)b(Y-EY)]$$

$$= a^2 \overbrace{E[(X-EX)^2]}^{\operatorname{Var}(X)} + b^2 \overbrace{E[(Y-EY)^2]}^{\operatorname{Var}(Y)} + 2ab \overbrace{E[(X-EX)(Y-EY)]}^{\operatorname{Cov}(X,Y)}$$

Theorem: $\underline{Cov(aX+b, Y) = a\,Cov(X,Y)}$

Recall: $Var(aX) = a^2\,Var(X)$

Pf:

$$Cov(aX+b, Y) = E[(aX+b-E[aX+b])(Y-EY)]$$
$$= E[(aX+\cancel{b}-aEX-\cancel{b})(Y-EY)]$$
$$= a\,\underbrace{E[(X-EX)(Y-EY)]}_{Cov(X,Y)}$$

Corollaries:

1. $Cov(X, cY+d) = c\,Cov(X,Y)$

$$\boxed{\text{2) } Cov(aX+b, cY+d) = ac\,Cov(X,Y)}$$

Theorem:

$$Cor(aX+b, cY+d) = sgn(a)\,sign(c)\,Cor(X,Y)$$

$$\operatorname{Sgn}(x) = \begin{cases} +1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases}$$

$\underline{\text{Ex}}$ $\quad Cor(-5X, Y) = -Cor(X, Y)$

Pf. $a, c \neq 0$

$\underline{\text{note}}$: $x \neq 0$

$$\operatorname{Sgn}(x) = \frac{x}{|x|}.$$

$$Cor(aX+b, cY+d) = \frac{Cov(aX+b, cY+d)}{\sqrt{Var(aX+b)\,Var(cY+d)}}$$

$$= \frac{ac\;Cov(X,Y)}{\sqrt{a^2 Var(X)\, c^2 Var(Y)}}$$

$$= \underbrace{\frac{a}{\sqrt{a^2}}}_{\frac{a}{|a|}} \underbrace{\frac{c}{\sqrt{c^2}}}_{\frac{c}{|c|}} \underbrace{\frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}}_{Cor(X,Y)}$$

Theorem: $-1 \le \underline{Cor(X,Y)} \le 1$

pf $\underline{\text{WLOG}}$ assume

$$EX = EY = 0$$

$$Var X = Var Y = 1$$

Notice: $Cor(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{\underbrace{Var(X)}_{1}\underbrace{Var(Y)}_{1}}} = Cov(X,Y)$

$$Var(X \pm Y) = \underbrace{Var(X)}_{1} + \underbrace{Var(Y)}_{1} \pm 2\underbrace{Cov(X,Y)}_{Cor(X,Y)}$$

$$= \underbrace{2 \pm 2Cor(X,Y) \geq 0}$$

$$\Rightarrow 1 \pm Cor(X,Y) \geq 0$$

$1 + Cor \geq 0$

$\boxed{Cor \geq -1}$

$1 - Cor \geq 0$

$\boxed{cor \leq 1}$

Variance Short-cut!

$$Var(X) = E[X^2] - E[X]^2$$

Covariance Short-Cut

$$Cov(X, Y) = E[XY] - E[X]E[Y].$$