Lecture 18

Short-cut for Covariance:

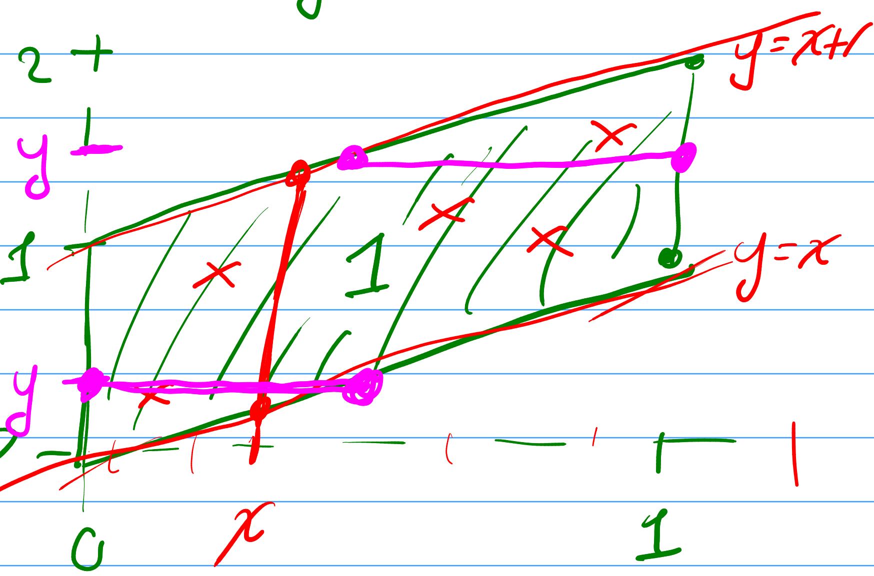
$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

Ex. $f(x,y) = 1 \qquad 0 < x < 1$
$\qquad\qquad\qquad\quad x < y < x+1$

Had calculated:

$$E[XY] = 7/12$$

What's the cov/cor?



Marginal of X:

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_x^{x+1} 1\,dy$$

$$\boxed{\begin{array}{c} = 1 \\ \text{for } 0 < x < 1 \end{array}}$$

$X \sim U(0,1)$

$E[X] = \frac{1}{2}$, $Var(X) = 1/12$.

Marginal of Y

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx =$$

$$= \begin{cases} \int_0^y 1 \, dx & 0<y<1 \\ \int_{y-1}^1 1 \, dx & 1<y<2 \end{cases}$$

$$= \begin{cases} y & 0<y<1 \\ 2-y & 1<y<2 \end{cases}$$

Can show:

$E[Y] = 1$

$Var(Y) = 1/6$

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$$
$$= 7/12 - \left(\tfrac{1}{2}\right)(1) = \frac{1}{12}$$

$$\text{Cor}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

$$= \frac{1/12}{\sqrt{\frac{1}{12}\cdot\frac{1}{6}}} \approx .7$$

Conditional Prob:

$$P(A|B) = \frac{P(AB)}{P(B)}$$

If X and Y are RVs let

$$A = \{X = x\} \text{ and } B = \{Y = y\}$$

then

$$P(A|B) = P(X = x \mid Y = y)$$

$$= P(AB)/P(B)$$

$$= P(X=x, Y=y)/P(Y=y)$$

$$= \boxed{f(x,y)/f_Y(y)}.$$

Defn: Conditional PMF

If X and Y are discrete RVs then the conditional PMF of X given $Y=y$ is

$$f(x|y) = f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}.$$

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can think of as a univariate RV

Ex. Joint PMF

$Y \backslash X$	10	20	30
2	0	0	4/18
1	3/18	4/18	3/18
0	2/18	2/18	0

$f_Y(0) = 4/18$

What's the dist of $X \mid Y = 0$

$$f(x|0) = \frac{f(x,0)}{f_Y(0)} = \begin{cases} \frac{2/18}{4/18} = \frac{1}{2} & , x = 10 \\ 1/2 & , x = 20 \\ 0 & , x = 30 \end{cases}$$

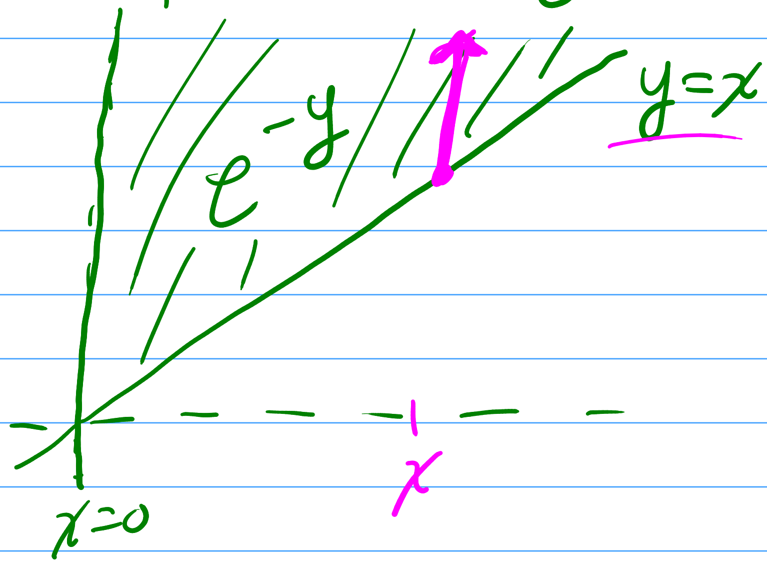
Defn: Conditional PDF

If X and Y are cts RVs then the conditional PDF of X given $Y=y$ is

$$f(x|y) = \frac{f(x,y)}{f_Y(y)}$$

← ratio of densities

Ex. $f(x,y) = e^{-y}$ for $0 < x < y$



What's the PDF of

$\boxed{Y \mid X = x}$

$$\boxed{f(y|x) = \frac{f(x,y)}{f_X(x)}}$$

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_x^\infty e^{-y}\,dy$$

$$= -e^{-y}\Big|_x^\infty$$

$$= 0 + e^{-x} = \boxed{e^{-x} \text{ for } x > 0}$$

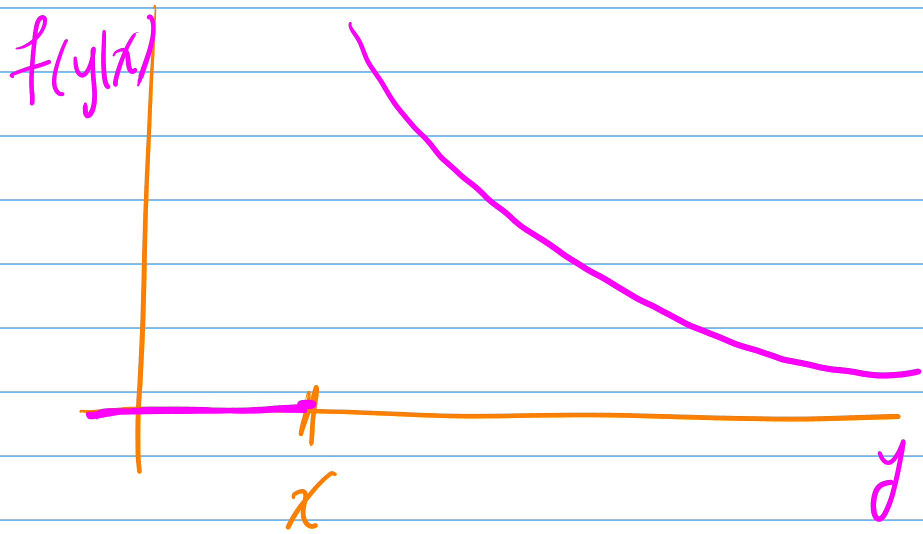
So . . .

$X \sim \text{Exp}(1)$

$$f(y|x) = \frac{e^{-y}}{e^{-x}} \quad \text{for } 0 < x < y$$

$$= e^{-(y-x)}$$

Called a shifted Exp dist.



Defn: Conditional Expectation

If g is a fn $g: \mathbb{R} \to \mathbb{R}$ then the cond. expect. of $g(X)$ given $Y = y$

$$\underbrace{E[g(X) \mid Y = y]}_{Z} = \begin{cases} \sum_x g(x) f(x|y) & \text{discrete} \\ \int_{\mathbb{R}} g(x) f(x|y)\, dx & \text{cts} \end{cases}$$

Ex. $f(x,y) = e^{-y}$ for $0 < x < y$

Shown: $f(y|x) = e^{-(y-x)}$ for $0 < x < y$

$$E[Y|X=x] = \int_{\mathbb{R}} y\, f(y|x)\, dy$$

$$= \int_x^\infty y\, e^{-(y-x)}\, dy$$

$$= \cdots = 1 + x$$

Defn: Conditional Variance

$$\mathrm{Var}(X|Y=y)$$

$$= E\left[\left(X - E[X|Y=y]\right)^2 \,\middle|\, Y=y\right]$$

Short-cut:

$$\mathrm{Var}(X|Y=y) = E[X^2|Y=y] - E[X|Y=y]^2.$$

Ex. Continue previous

$$E[Y^2 | X = x] = \int_{\mathbb{R}} y^2 f(y|x)\, dy$$

$$= \int_x^{\infty} y^2 e^{-(y-x)}\, dy$$

$$= \cdots = x^2 + 2x + 2$$

$$\mathrm{Var}(Y | X = x) = (x^2 + 2x + 2) - (1 + x)^2$$

$$= \cdots$$

$$= 1$$

Independence:

For events: If $A, B \subset S$ then

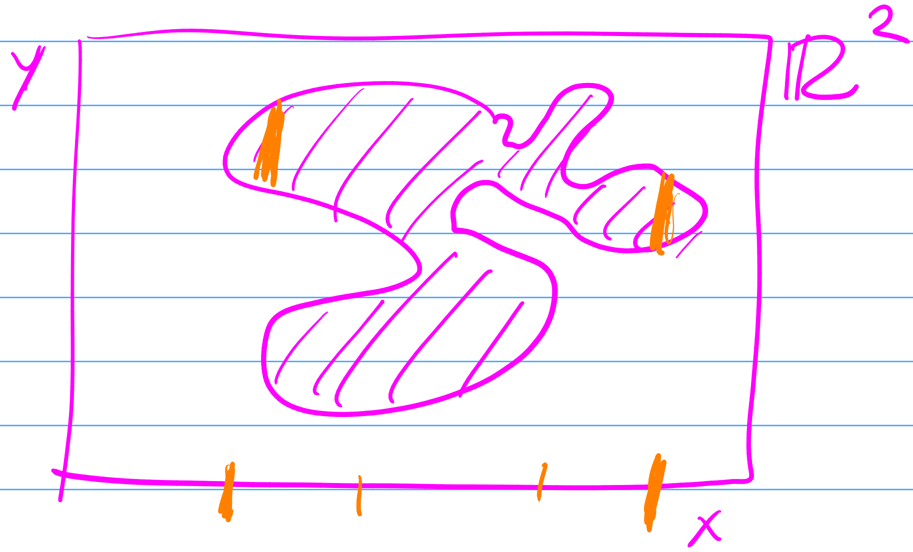
$$A \perp B \text{ if } P(AB) = P(A)P(B).$$

For RVs:

$$X \perp Y \text{ if } P(X \in A, Y \in B) = P(X \in A) P(Y \in B) \quad \forall A, B \subset \mathbb{R}$$

Product Space:

Support $(X, Y) = \{(x,y) \mid f(x,y) > 0\}$

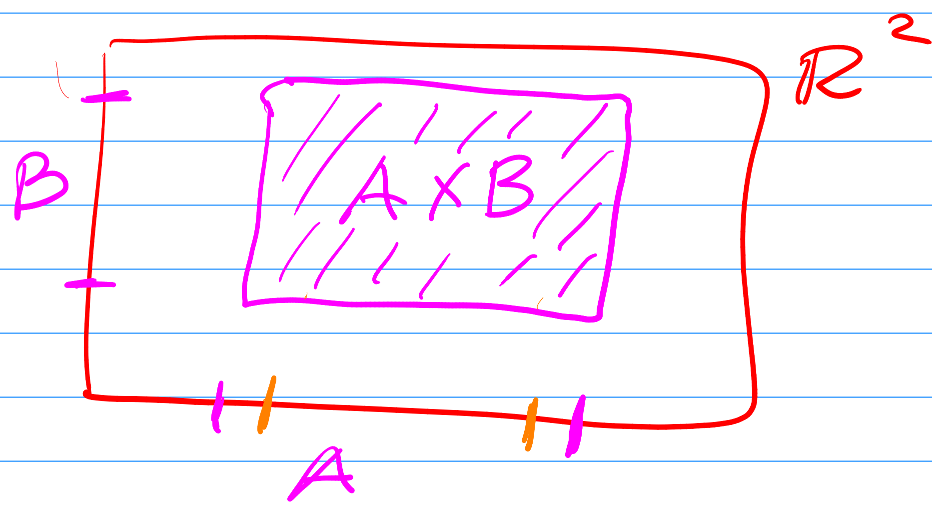


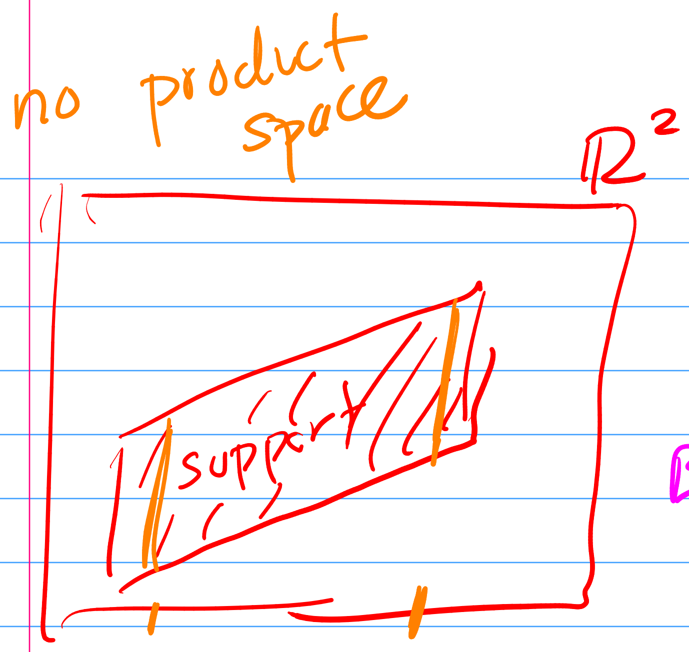
Often: $f(x,y) =$ ~~~ $\boxed{\text{for } x \in A,\ y \in B}$

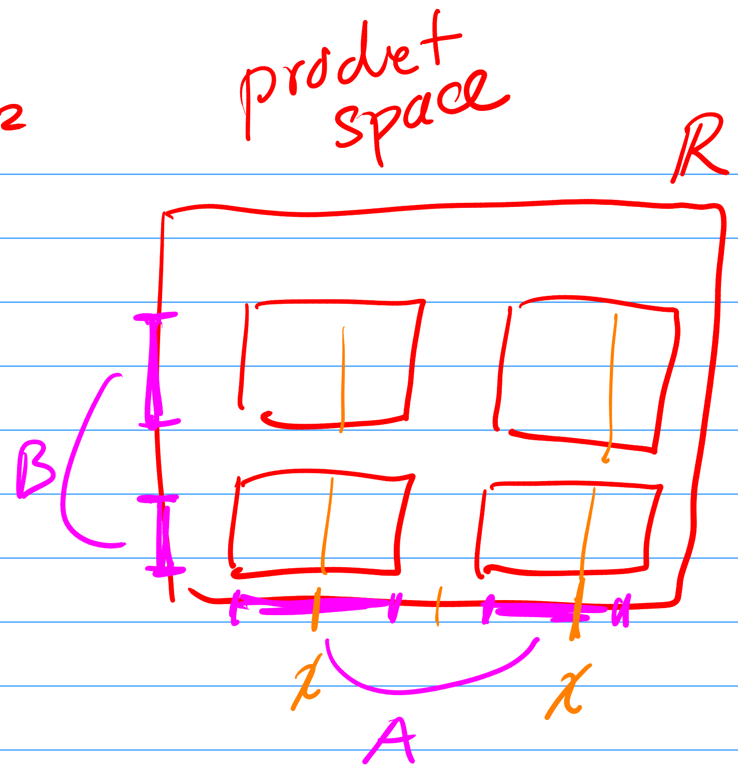
no y   no x

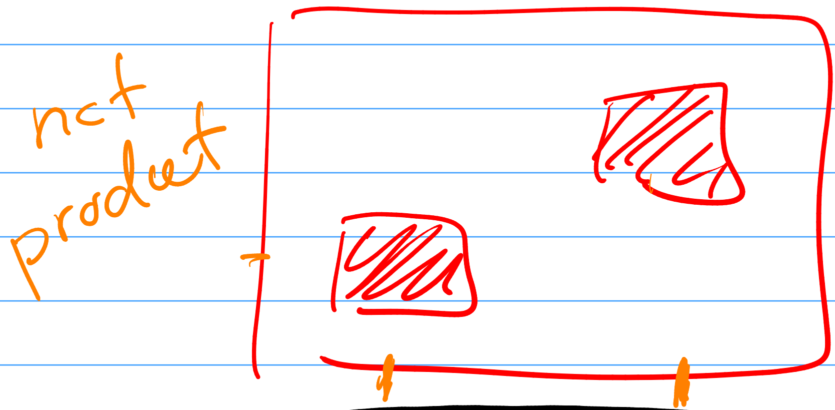
then support is the product $A \times B$

$= \{(x,y) \mid x \in A,\ y \in B\}$









Theorem: Factorization Theorem

$X \perp Y$

iff (1) Support of X, Y is a product space

AND

(2) either

$$F(x,y) = F_X(x) F_Y(y)$$

or $$f(x,y) = f_X(x) f_Y(y)$$

Ex, $f(x,y)$

$y \backslash x$	10	20	f_Y
f_X	$1/2$	$1/2$	
3	$1/5$	$3/10$	$1/2$
2	$1/5$	$1/10$	$3/10$
1	$1/10$	$1/10$	$1/5$

Q: Are X and Y independent?

(1) Product space? $\{10, 20\} \times \{1, 2, 3\}$

(2) $f(x,y) = f_X(x) f_Y(y)$?

Ex: $f(10,3) = f_X(10) f_Y(3)$?

$$\frac{1}{5} \neq \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

So not independent.