Lecture 18

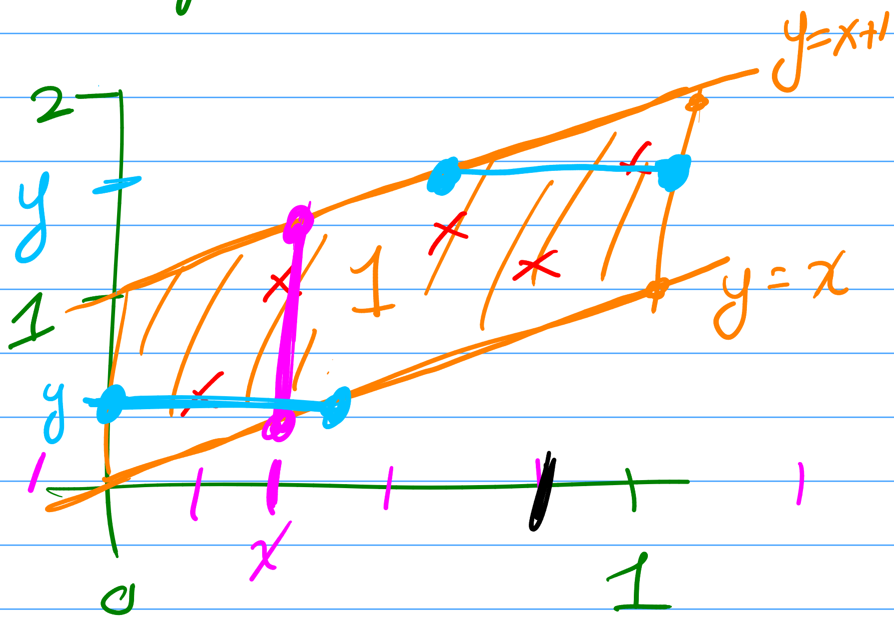
$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

Ex. $f(x,y) = 1 \quad 0 < x < 1, \quad x < y < x+1$

Had shown:

$E[XY] = 7/12$

What's the cov/cor?



Marginal of X

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_x^{x+1} 1\,dy$$

$$= (x+1) - x = 1$$

$$\boxed{= 1 \quad \text{For } 0 < x < 1}$$

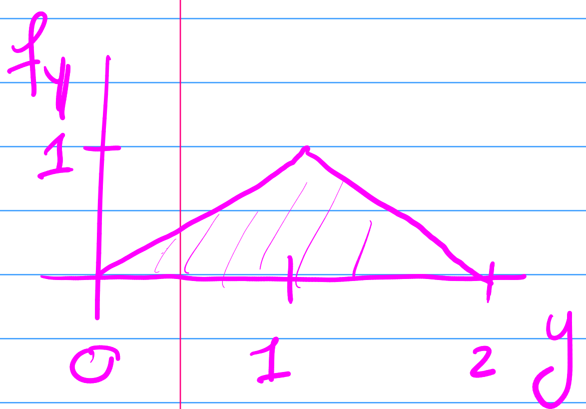
So $X \sim U(0,1)$

$$E[X] = 1/2 \ , \quad Var(X) = 1/12$$

Marginal of Y

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \begin{cases} \int_0^y 1\,dx & 0<y<1 \\ \int_{y-1}^1 1\,dx & 1<y<2 \end{cases}$$

$$= \begin{cases} y & 0<y<1 \\ 2-y & 1<y<2 \end{cases}$$



Can show:

$$E[Y] = 1$$

$$Var(Y) = 1/6$$

$$Cov(X,Y) = E[XY] - E(X)E(Y)$$

$$= 7/12 - \left(\tfrac{1}{2}\right)(1) = 1/12$$

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

$$= \frac{1/12}{\sqrt{\frac{1}{12} \cdot \frac{1}{6}}} \approx .7$$

Conditional Probability

$$P(A|B) = \frac{P(AB)}{P(B)}$$

If X and Y are discrete then

$$A = \{X = x\} \text{ and } B = \{Y = y\}$$

then

$$P(X=x|Y=y) = P(A|B) = \frac{P(AB)}{P(B)}$$

$$= P(X=x, Y=y) / P(Y=y)$$

$$= \left[f(x,y) / f_Y(y) \right.$$

Defn: Conditional PMF

If X and Y are discrete then the conditional PMF of X given $Y=y$ is

$$f(x|y) = f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$$

a univariate RV

Ex. Joint PMF

Y \ X	10	20	30
2	0	0	4/18
1	3/18	4/18	3/18
0	2/18	2/18	0

$f_Y(0) = \frac{4}{18}$

what's the dist of $X | Y = 0$

$$f(x|0) = \frac{f(x,0)}{f_Y(0)} = \begin{cases} \frac{2/18}{4/18} = \frac{1}{2}, & x=10 \\ \frac{2/18}{4/18} = \frac{1}{2}, & x=20 \\ 0/4/18 = 0, & x=30 \end{cases}$$

$$= \begin{cases} \frac{1}{2}, & x=10 \\ \frac{1}{2}, & x=20 \end{cases}$$

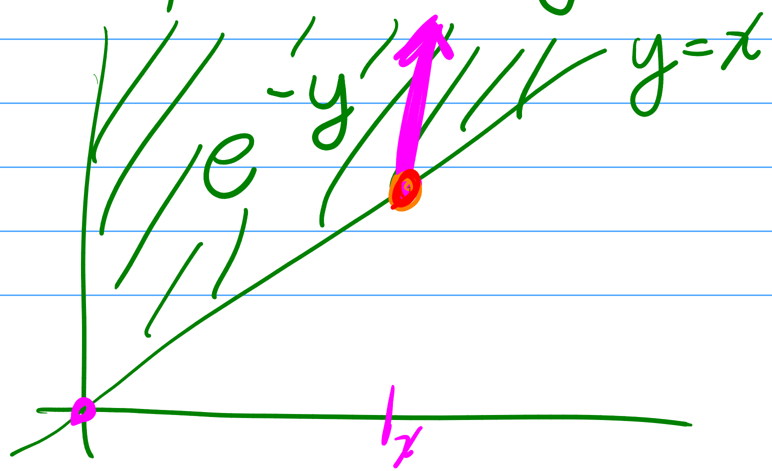
Defn: Conditional PDF

If X and Y are cts then the cond. PDF of X given $Y=y$ is

$$f(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

Ex. $f(x,y) = e^{-y}$ for $0<x<y$

What's the PDF of $Y|X=x$.



$$f(y|x) = \frac{f(x,y)}{f_X(x)}$$

Get marginal of X:

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\, dy$$

$$= \int_x^\infty e^{-y}\, dy = \left[-e^{-y}\right]_x^\infty$$

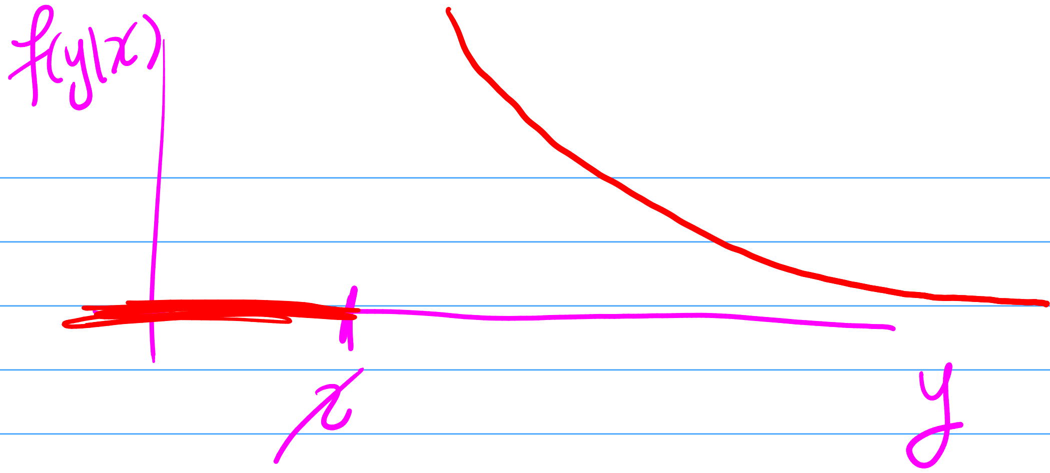
$$= 0 + e^{-x}$$

$$\boxed{= e^{-x} \text{ for } x > 0}$$

$X \sim \text{Exp}(1)$

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} \text{ for } 0<x<y$$

$$\boxed{= e^{-(y-x)} \quad \text{for } 0<x<y}$$



Called: Shifted Exp.

Defn: Conditional Expectation

If $g: \mathbb{R} \to \mathbb{R}$ then the conditional expect. of $g(X)$ given $Y=y$ is

$$E[g(X) \mid Y=y] = \begin{cases} \sum_x g(x) f(x|y) & \text{discrete} \\ \int_{\mathbb{R}} g(x) f(x|y)\,dx & \text{cts} \end{cases}$$

Ex, Continue prev.

$$E[Y \mid X=x] = \int_{\mathbb{R}} y\, f(y|x)\,dy$$

$$= \int_x^\infty y e^{-(y-x)} dy$$

$$= \cdots = 1+x$$

Defn: Conditional Variance

$$\mathrm{Var}(X \mid Y=y)$$

$$= E\left[\left(X - E[X \mid Y=y]\right)^2 \mid Y=y\right]$$

NB: $\mathrm{Var}(X) = E[(X - EX)^2]$

Short-cut:

$$\mathrm{Var}(X \mid Y=y) = E[X^2 \mid Y=y] - E[X \mid Y=y]^2.$$

Ex. Continue prev.

$$E[Y^2 \mid X=x] = \int y^2 f(y \mid x)\, dy$$

$$= \int_x^\infty y^2 e^{-(y-x)} dy$$

$= \dots$

$= x^2 + 2x + 2$

$$Var(Y \mid X = x) = (x^2 + 2x + 2) - (x+1)^2$$

$$= 1$$

Independence for Events

$$A \perp B \iff P(AB) = P(A)P(B)$$

For RVs

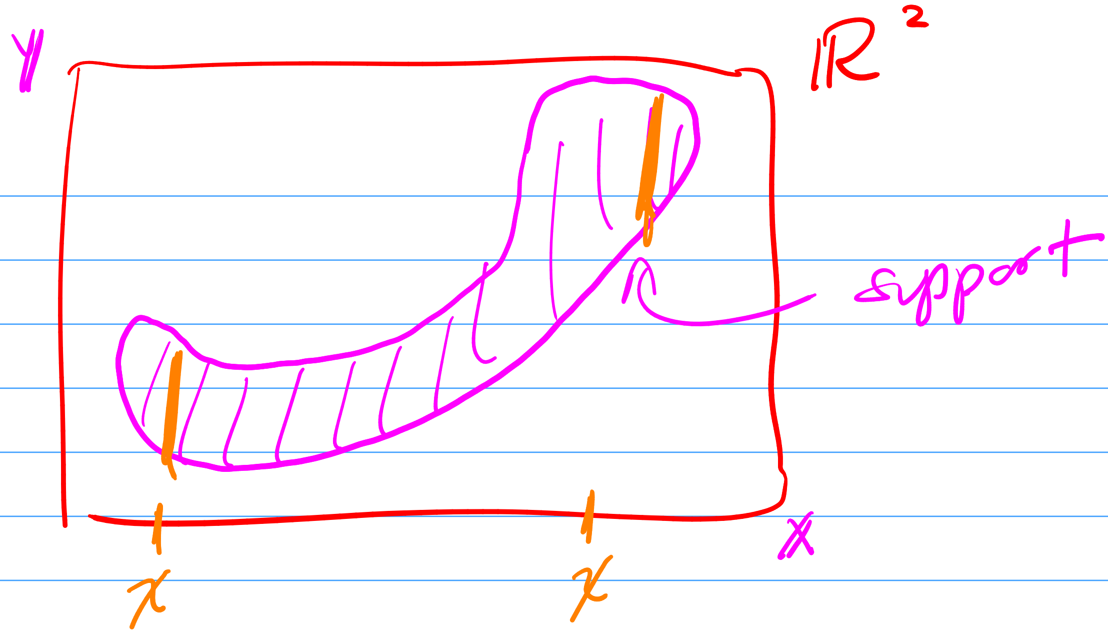
$$X \perp Y \iff P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

$$\forall A, B \subset \mathbb{R}$$

Product Spaces

$\subset \mathbb{R}^2$

$$\text{Support}(X, Y) = \{(x,y) \mid f(x,y) > 0\}$$



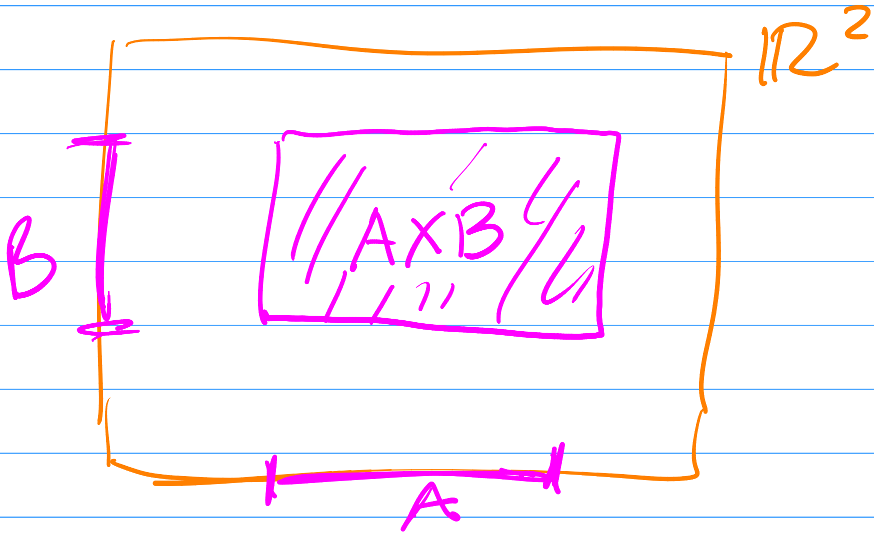
often:

$$f(x,y) = \sim\sim\sim \text{ for } x \in A, \; y \in B$$

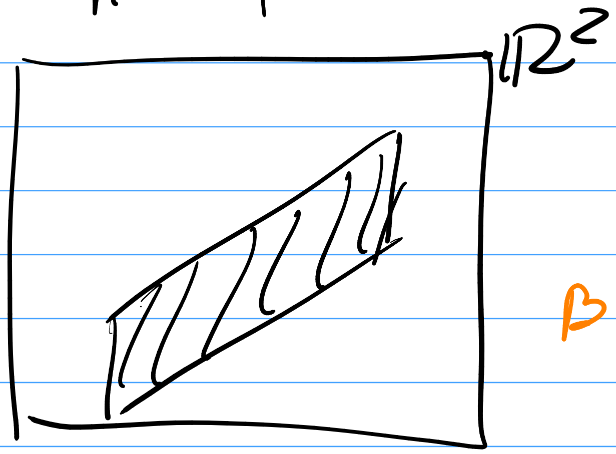
no y   no x

the support is

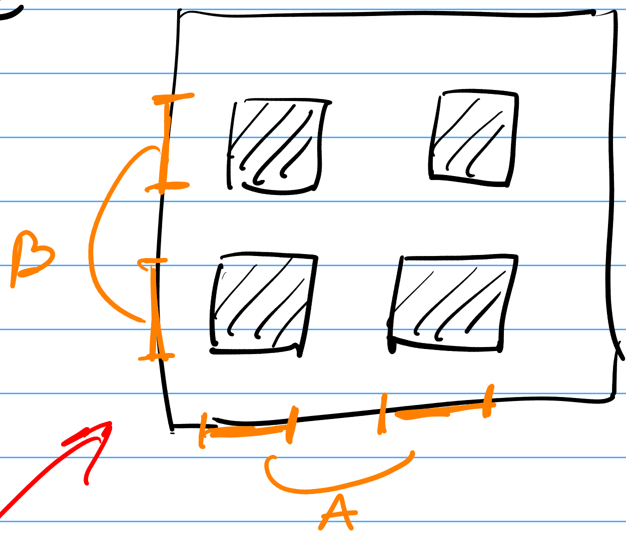
$$A \times B = \{(x,y) \mid x \in A, y \in B\}$$



Ex, not product



is product space



$$A = [0,1] \cup [2,3]$$

$$B = [-1,1] \cup [5,6]$$

$$A \times B$$

Theorem: **Factorization Theorem**

$X \perp Y$ iff

(1) support is a product space

AND

(2) $f(x,y) = f_X(x) f_Y(y)$

$[\text{or } F(x,y) = F_X(x) F_Y(y)]$