# Lecture 19

Ex, $f(x,y) = \frac{1}{384} x^2 e^{-y-(x/2)}$

for $x>0, y>0$

$X \perp Y$?

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Corollary:

$X \perp Y$ iff

1. Support is a product space
2. $f(x,y) = g(x)\, h(y)$

no $y$   no $x$

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Ex, continue

1. is support product space?

Yes : $(0,\infty)\times(0,\infty)$

② 

$$f(x,y) = \frac{1}{384} x^2 e^{-y-(x/2)}$$

$$= \frac{1}{384} x^2 e^{-y} e^{-(x/2)}$$

$$= \underbrace{\frac{1}{384} x^2 e^{-(x/2)}}_{g(x)} \underbrace{e^{-y}}_{h(y)}$$

So $X \perp Y$.

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<u>For events</u>: $A \perp B$ then $P(A|B) = P(A)$

<u>For RVs</u>: If $X \perp Y$ then

$$f(x|y) = f_X(x)$$

<u>pf.</u> $f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x) f_Y(y)}{f_Y(y)}$

$$= f_X(x)$$

Theorem:

If $X \perp Y$ and $g_1 : \mathbb{R} \to \mathbb{R}$, $g_2 : \mathbb{R} \to \mathbb{R}$

$$E[g_1(X) g_2(Y)] = E[g_1(X)] E[g_2(Y)]$$

Pf. (cts)

$$E[g_1(X) g_2(Y)] = \iint_{A \times B} g_1(x) g_2(y) f(x,y) \, dx \, dy$$

$$= \int_B \int_A g_1(x) g_2(y) f_X(x) f_Y(y) \, dx \, dy$$

$$= \left[ \int_A g_1(x) f_X(x) \, dx \right] \left[ \int_B g_2(y) f_Y(y) \, dy \right]$$

$$= E[g_1(X)] E[g_2(Y)]$$

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Ex. $X, Y \overset{iid}{\sim} \text{Exp}(1)$

independent and identically distributed

$$E[X^2Y] = E[X^2]E[Y]$$
$$= (2)(1)$$
$$= 2$$

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Theorem: MGF of Sum of Independent

If $X \perp Y$ then

$$M_{X+Y}(t) = M_X(t)M_Y(t)$$

pf.

$$M_{X+Y}(t) = E[e^{t(X+Y)}]$$
$$= E[e^{tX}e^{tY}]$$
$$= E[e^{tX}]E[e^{tY}]$$
$$= M_X(t)\,M_Y(t)$$

Theorem:

$X \sim N(\mu, \sigma^2)$

$Y \sim N(\gamma, \tau^2)$

and $X \perp Y$.

then $X+Y \sim N(\mu+\gamma, \sigma^2+\tau^2)$.

pf. $M_{X+Y}(t) = M_X(t) M_Y(t)$

$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \exp\left(\gamma t + \frac{\tau^2 t^2}{2}\right)$$

$$= \exp\left((\mu+\gamma)t + \frac{(\sigma^2+\tau^2)t^2}{2}\right)$$

$\boxed{\exp\left((\text{mean})t + \frac{(\text{Var})t^2}{2}\right)}$

MGF of a $N(\mu+\gamma, \sigma^2+\tau^2)$

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Theorem:

If $X \perp Y$ then $Cov(X, Y) = Cor(X, Y) = 0$.

Pf.

$$\begin{aligned} Cov(X,Y) &= E[XY] - E[X]E[Y] \\ &= E[X]E[Y] - E[X]E[Y] \\ &= 0 \end{aligned}$$

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Generally, converse is false:

If $Cor(X,Y) = 0$, could be indep. or not.

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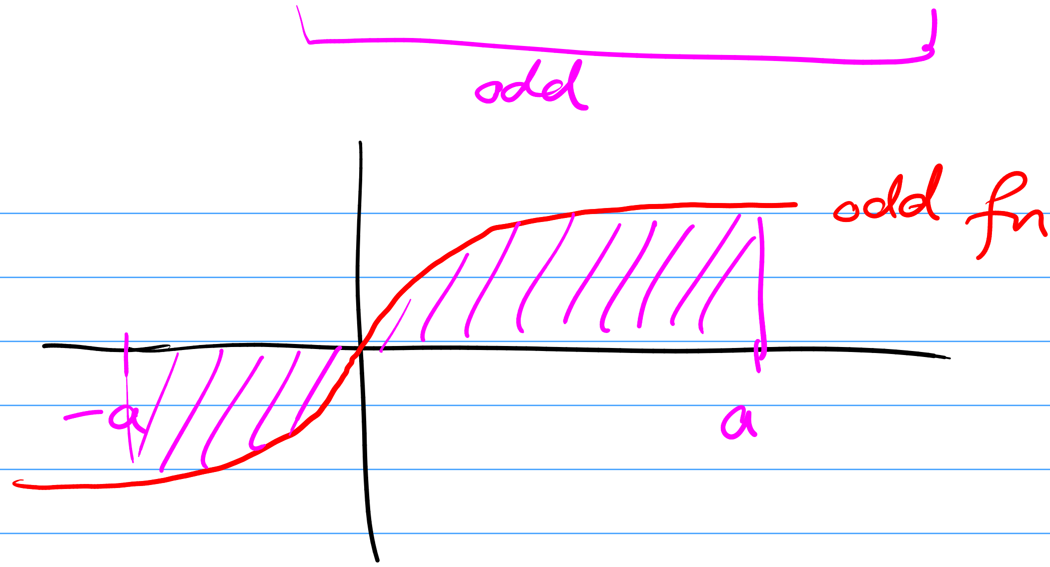
Ex. $X \sim N(0,1)$ and $Y = X^2$.

not $\perp$

$$\begin{aligned} Cov(X,Y) &= E[XY] - E[X]E[Y] \\ &= E[XX^2] - \underbrace{E(X)}_{0} E[X^2] \\ &= E[X^3] \\ &= \int_{\mathbb{R}} \underbrace{x^3}_{odd} \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)}_{even} dx = 0 \end{aligned}$$




So $Cor(X, Y) = 0$

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<u>For events:</u> Bayes' Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

<u>For RVs:</u>

$$f(x|y) = \frac{f(y|x) f_X(x)}{f_Y(y)}.$$

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<u>For events:</u> Total Prob.

If $C_i$ partition $S$ then

$$P(A) = \sum_i P(A|C_i)P(C_i)$$

**RV:**

**discrete:** $f(y) = \sum_x f(y|x) f(x)$

**cts:** $\boxed{f(y) = \int_{\mathbb{R}} f(y|x) f(x)\, dx}$

Pf. (cts)

① $f(y|x) = \dfrac{f(x,y)}{f(x)} \Leftrightarrow f(x,y) = f(y|x) f(x)$

② $f(y) = \int f(x,y)\, dx$

$= \int f(y|x) f(x)\, dx$ .

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**Ex.** $X \sim \text{Exp}(\lambda)$

$Y | X = x \sim \text{Pois}(x)$

what's the dist of $Y$?

$$f(y) = \int_{\mathbb{R}} \overbrace{f(y|x)}^{\text{Pois}(x)} \overbrace{f(x)}^{\text{Exp}(\lambda)} dx$$

$$= \underbrace{\int_0^\infty \frac{x^y e^{-x}}{y!}}_{\text{Pois}(x)} \underbrace{\lambda e^{-\lambda x}}_{\text{Exp}(\lambda)} dx$$

$$= \frac{\lambda}{y!} \int_0^\infty x^y e^{-(\lambda+1)x} dx$$

$z \sim \text{Pois}(\mu)$

$$f(z) = \frac{\mu^z e^{-\mu}}{z!}$$

PDF Gamma$(a,b)$

$$x^{a-1} e^{-bx} \frac{b^a}{\Gamma(a)}$$

$y = a-1 \Rightarrow a = y+1$

$\lambda+1 = b$

$$= \frac{\lambda}{y!} \frac{\Gamma(y+1)}{(\lambda+1)^{y+1}} \underbrace{\int_0^\infty x^y e^{-(\lambda+1)x} \frac{(\lambda+1)^{y+1}}{\Gamma(y+1)} dx}_{1}$$

$$= \frac{\lambda}{y!} \frac{\overbrace{\Gamma(y+1)}^{y!}}{(\lambda+1)^{y+1}}, \quad y = 0, 1, 2, 3, \ldots$$

$$\boxed{f(y) = \frac{\lambda}{(\lambda+1)^{y+1}}, \quad y = 0, 1, 2, \ldots}$$

Ex. $Y \sim Pois(\lambda)$

$X \mid Y = y \sim Bin(y, p)$ ← Known, $p \in [0,1]$

note: $X \le Y$

What's the dist of $X$?

$$f(x) = \sum_y \underbrace{f(x|y)}_{Bin(y,p)} \underbrace{f(y)}_{Pois(\lambda)}$$

$$= \sum_{y=x}^{\infty} \binom{y}{x} p^x (1-p)^{y-x} \frac{\lambda^y e^{-\lambda}}{y!} \dots$$