Lecture 19

Joint PMF

<u>Ex.</u>

y \ x	10	20	f_Y
3	1/5	3/10	1/2
2	1/5	1/16	3/10
1	1/10	1/10	2/10
f_X	1/2	1/2	

Are X and Y indep?

(1) <u>Product Space?</u>

Support $= \{10, 20\} \times \{1, 2, 3\}$

(2) <u>Factor?</u> $f(x,y) = f_X(x) f_Y(y)$

$$f(10,2) = \frac{1}{5} \cancel{\overset{?}{=}} \left(\frac{1}{2}\right)\left(\frac{3}{10}\right) = f_X(10) f_Y(2)$$

Not independent.

Ex.

$$f(x,y) = \frac{1}{384} x^2 e^{-y-(x/2)}$$

$$\boxed{\text{for } x>0,\ y>0}$$

Corollary:

$X \perp Y$ iff

(1) Support is product space

(2) $f(x,y) = g(x)h(y)$

no y no x

Ex. Continue

(1) Yes, product space:

Support $= (0,\infty) \times (0,\infty)$

(2) $f(x,y) = \frac{1}{384} x^2 e^{-y-x/2}$

$= \frac{1}{384} x^2 e^{-y} e^{-x/2}$

$= \underbrace{\frac{1}{384} x^2 e^{-x/2}}_{g(x)} \underbrace{e^{-y}}_{h(y)}$

Theorem:

If $X \perp Y$ and $g_1 : \mathbb{R} \to \mathbb{R}$, $g_2 : \mathbb{R} \to \mathbb{R}$

then

$$E[g_1(X) g_2(Y)] = E[g_1(X)] E[g_2(Y)]$$

Pf (cts)

$$E[g_1(X) g_2(Y)] = \iint_{A \times B} g_1(x) g_2(y) \underline{f(x,y)}\, dx\, dy$$

$$= \int_B \int_A g_1(x) g_2(y) f(x) f(y)\, dx\, dy$$

$$= \left[\int_A g_1(x) f(x)\, dx \right] \left[\int_B g_2(y) f(y)\, dy \right]$$

$$= E[g_1(X)] E[g_2(Y)]$$

Ex $X, Y \overset{iid}{\sim} \text{Exp}(1)$

independent, identically distributed

$$E[X^2Y] = E[X^2]E[Y]$$
$$= (2)(1) = 2$$

Theorem:

$Z = X + Y$

If $X \perp Y$ then

$$M_{X+Y}(t) = M_X(t)\, M_Y(t).$$

pf.

$M_Z(t) =$

$$M_{X+Y}(t) = E\left[e^{t(X+Y)}\right]$$
$$= E\left[e^{tX}e^{tY}\right]$$
$$= E\left[e^{tX}\right]E\left[e^{tY}\right]$$
$$= M_X(t)\, M_Y(t)$$

Theorem:

$$\left.\begin{array}{l} X \sim N(\mu, \sigma^2) \\ Y \sim N(\gamma, \tau^2) \end{array}\right\} \text{ assume } X \perp Y.$$

$X + Y \sim N(\mu+\gamma,\ \sigma^2+\tau^2)$.

Pf.

$$M_{X+Y}(t) = M_X(t)\,M_Y(t)$$

$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\exp\left(\gamma t + \frac{\tau^2 t^2}{2}\right)$$

$$= \exp\left((\mu+\gamma)t + \frac{(\sigma^2+\tau^2)t^2}{2}\right)$$

MGF of $N(\mu+\gamma, \sigma^2+\tau^2)$

Theorem:

If $X \perp Y$ then $\mathrm{Cor}(X,Y) = 0 = \mathrm{Cov}(X,Y)$

Pf.

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$= E[X]E[Y] - E[X]E[Y] = 0$$

Generally, converse is false.

If $\mathrm{cov}(X,Y)=0$ then X and Y could be dependent.

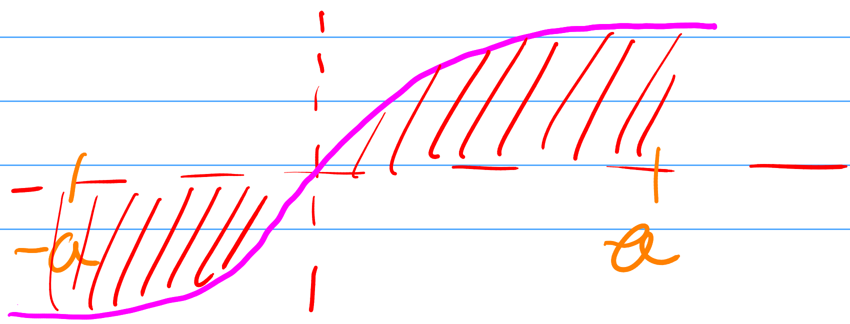
Ex, $X \sim N(0,1)$, $Y = X^2$. $\Big\}$ not independent

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$= E[X^3] - \underbrace{E[X]}_{0}E[X^2]$$

$$= E[X^3]$$

$$= \int_{\mathbb{R}} \underbrace{\underbrace{x^3}_{\text{odd}}\,\underbrace{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)}_{\text{even}}}_{\text{odd}}\,dx = 0$$



For events: $A \perp B$ then $P(A|B) = P(A)$

For RVs: $X \perp Y$ then $f(x|y) = f(x)$

pf. $f(x|y) = \dfrac{f(x,y)}{f(y)} = \dfrac{f(x)\cancel{f(y)}}{\cancel{f(y)}} = f(x)$

Bayes' Theorem:

For events: $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

For RVs: $f(x|y) = \dfrac{f(y|x)f(x)}{f(y)}$.

Partitioning (Total Prob.)

For events: C_i partitioned S

then $$P(A) = \sum_i P(A|C_i)P(C_i)$$

For RVs:

discrete : $f(y) = \sum_x f(y|x) f(x)$

cts: $f(y) = \int_{\mathbb{R}} f(y|x) f(x)\, dx$

pf. (cts)

1. $f(y|x) = \dfrac{f(x,y)}{f(x)} \Leftrightarrow f(x,y) = f(y|x) f(x)$

2. $f(y) = \int_{\mathbb{R}} f(x,y)\, dx$

$$\boxed{= \int_{\mathbb{R}} f(y|x) f(x)\, dx}$$

Ex. $\boxed{X \sim Exp(\lambda)}$

$Y | X = x \sim Pois(x)$

what's the dist of Y?

$$f(y) = \int \overbrace{f(y|x)}^{Pois(x)} \overbrace{f(x)}^{Exp(\lambda)} dx$$

$$= \int_0^\infty \frac{x^y e^{-x}}{y!} \lambda e^{-\lambda x} dx$$

$$= \frac{\lambda}{y!} \int_0^\infty x^{y} e^{-(\lambda+1)x} dx$$

$Z \sim Pois(\mu)$

$f(z) = \frac{\mu^z e^{-\mu}}{z!}$

$Gamma(a, b)$

$PDF = x^{a-1} e^{-bx} \frac{b^a}{\Gamma(a)}$

$y = a-1 \Rightarrow \boxed{a = y+1}$

$\boxed{b = \lambda+1}$

$$= \frac{\lambda}{y!} \frac{\Gamma(y+1)}{(\lambda+1)^{y+1}} \underbrace{\int_0^\infty x^y e^{-(\lambda+1)x} \frac{(\lambda+1)^{y+1}}{\Gamma(y+1)} dx}_{\text{integrate to } 1}$$

$$= \frac{\lambda}{y!} \frac{\overbrace{\Gamma(y+1)}^{y!}}{(\lambda+1)^{y+1}} \quad \text{for } y = 0, 1, 2, \ldots$$

$$\boxed{f(y) = \frac{\lambda}{(\lambda+1)^{y+1}} \quad \text{for } y = 0, 1, 2, \ldots}$$

ex, $\boxed{Y \sim Pois(\lambda)}$ Known

$X \mid Y = y \sim Bin(y, p)$

note: $\boxed{X \leq Y}$

What's the dist of X?

$$f(x) = \sum_y \underbrace{f(x|y)}_{Bin(y,p)} \underbrace{f(y)}_{Pois\,\lambda}$$

$X \sim Bin(n,p)$

$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$

$$= \sum_{y=x}^{\infty} \binom{y}{x} p^x (1-p)^{y-x} \frac{\lambda^y e^{-\lambda}}{y!}$$