Lecture 20

$Y \sim \text{Pois}(\lambda)$

$X \mid Y = y \sim \text{Bin}(y, p)$

note: $X \le Y$

What is the dist of X?

$$f(x) = \sum_y \underbrace{f(x|y)}_{\text{Bin}(y,p)} \underbrace{f(y)}_{\text{Pois}(\lambda)}$$

$$= \sum_{y=x}^{\infty} \binom{y}{x} p^x (1-p)^{y-x} \frac{\lambda^y e^{-\lambda}}{y!}$$

Bin(n,p)

PDF:

$\binom{n}{x} p^x (1-p)^{n-x}$

$$= \frac{p^x e^{-\lambda} \lambda^x}{x!} \sum_{y=x}^{\infty} \frac{1}{(y-x)!} (1-p)^{y-x} \lambda^{y-x}$$

$$\frac{y!}{x!(y-x)!} \frac{1}{y!} = \frac{1}{x!(y-x)!}$$

$$= \frac{\lambda^x p^x e^{-\lambda}}{x!} \sum_{y=0}^{\infty} \frac{1}{y!} [\lambda(1-p)]^y$$

$$e^z = \sum_{i=0}^{\infty} z^i / i!$$

$$= \frac{\lambda^x p^x e^{-\lambda}}{x!} e^{\lambda(1-p)}$$

$$= \frac{(p\lambda)^x e^{-\lambda p}}{x!} = f(x)$$

$\hookleftarrow$ Pois$(p\lambda)$

$X \sim \text{Pois}(p\lambda)$

Theorem: Iterated Expectation

If X and Y are RVs then

$$E[X] = E_Y\big[\underbrace{E[X \mid Y]}_{\text{a RV (a fn of } Y)}\big].$$

Prev: $E[X \mid Y=y] = \int_{\mathbb{R}} x f(x|y)\,dx$

$= $ a number

For each $y \in \mathbb{R}$ I get a different corresp val.

$$g(y) = \boxed{E[X \mid Y=y]} \text{ a number}$$

e.g. $g(y) = y^2$ or $g(y) = y+1$

Can plug Y into g

e.g. $g(Y) = Y^2$ or $g(Y) = Y+1$

$g(Y) = E[X \mid \cancel{Y = y}]$

weird notation

tend to use notation

$g(Y) = E[X \mid Y]$

e.g. $E[X|Y] = Y^2$ ← a RV.

<u>Ex.</u> Prev. $Y \sim Pois(\lambda)$

$X|Y=y \sim Bin(y,p)$

what is $E[X]$?

① $E[X \mid Y=y] = yp$

② $E[X|Y] = Yp$

③ $E[X] = E[E[X|Y]]$

$= E[Yp]$

$= pE[Y] = p\lambda.$

Ex. $P \sim Beta(\alpha, \beta)$

$X | P = p \sim Bin(n, p)$

$E[X]$?

① $E[X|P=p] = np$

② $E[X|P] = nP$

③ $E[X] = E[E[X|P]]$

$= E[nP]$

$= nE[P] = n\frac{\alpha}{\alpha+\beta}.$

pf of theorem (cts)

(1) $$\boxed{f(x) = \int_{\mathbb{R}} f(x,y)\,dy}$$

(2) $$f(x|y) = \frac{f(x,y)}{f(y)} \iff \boxed{f(x,y) = f(x|y)\,f(y)}$$

(3) $$\boxed{g(y) = E[X|Y=y] = \int_{\mathbb{R}} x\,f(x|y)\,dx}$$

$$E[X] = \int_{\mathbb{R}} x\,\underbrace{f(x)}_{(1)}\,dx$$

$$= \int_{\mathbb{R}} x \int_{\mathbb{R}} \underbrace{f(x,y)}_{(2)}\,dy\,dx$$

$$= \int_{\mathbb{R}} x \int_{\mathbb{R}} f(x|y)\,f(y)\,dy\,dx$$

$$= \int_{\mathbb{R}} \underbrace{\int_{\mathbb{R}} x\,f(x|y)\,dx}_{g(y)}\, f(y)\,dy$$

$$= \int_{\mathbb{R}} g(y) f(y)\, dy$$

$$= E[g(Y)]$$

defn

$$= E[E[X|Y]]$$

Theorem: Law of total Variance

$$Var(X) = E[\underbrace{Var(X|Y)}] + Var(\underbrace{E[X|Y]})$$

Similar defn

Ex. $P \sim Beta(\alpha, \beta)$

$X | P = p \sim Bin(n, p)$

$Var(X)$?

① $E[X \mid P=p] = np$

$Var(X \mid P=p) = np(1-p)$

② $E[X \mid P] = nP$

$Var(X \mid P) = nP(1-P)$

③ $Var(X) = E[Var(X \mid P)] + Var(E[X \mid P])$

$= E[nP(1-P)] + Var(nP)$

$= n(E[P] - E[P^2]) + n^2 Var(P)$

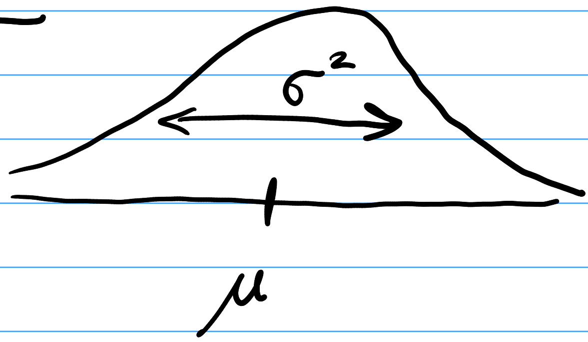
$\hookrightarrow Var(P) + E[P]^2$

$= \cdots$

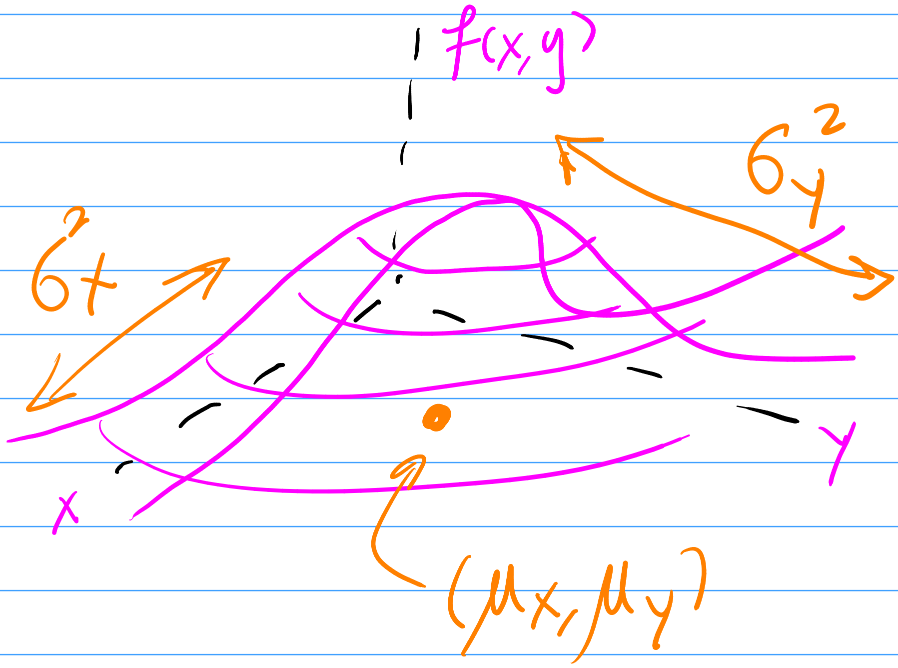
$$= n \frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)} + n^2 \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Bivariate Normal

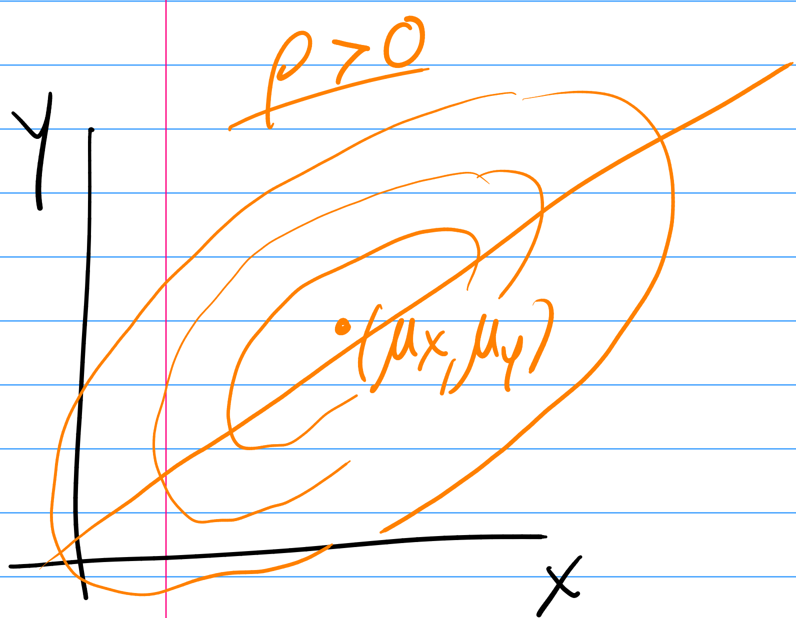
Uni: $N(\mu, \sigma^2)$

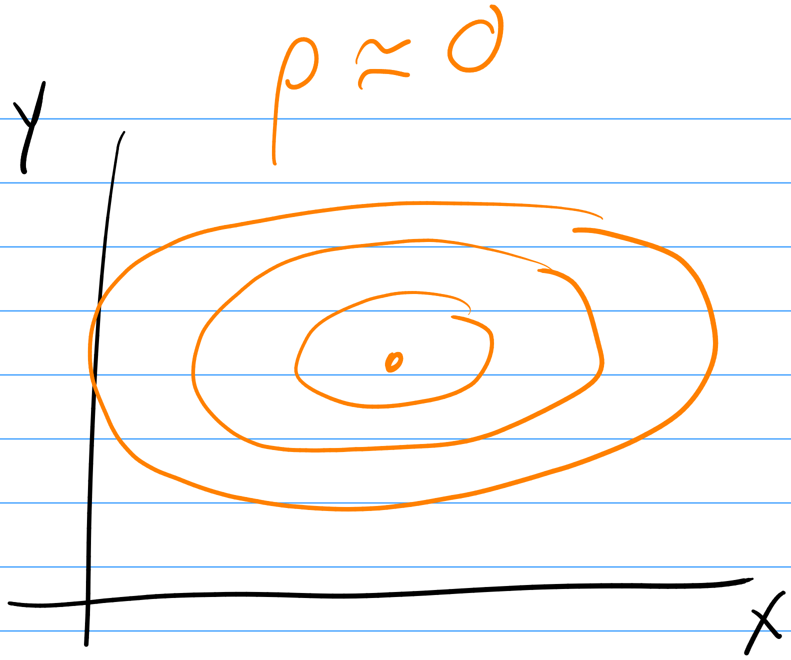


Bivariate:



need: correlation ρ





Density:

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot$$

$$\exp\left\{-\frac{1}{2\sqrt{1-\rho^2}}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]\right\}$$

alt. $\mu = (\mu_x, \mu_y)$ mean vector

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_x\sigma_y\rho \\ \sigma_x\sigma_y\rho & \sigma_y^2 \end{bmatrix} = \begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(Y,X) & Var(Y) \end{bmatrix}$$

Covariance matrix

$z = (x, y)$

$$f(z) = \underbrace{\frac{1}{2\pi}}\ \underbrace{\frac{1}{\sqrt{\det \Sigma}}}\ \exp\Big(-\frac{1}{2}\overbrace{\underbrace{(z-\mu)^T}_{1\times 2}\underbrace{\Sigma^{-1}}_{2\times 2}\underbrace{(z-\mu)}_{2\times 1}}^{\text{a number}}\Big)$$

uni:

$$f(x) = \frac{1}{\sqrt{2\pi}}\ \frac{1}{\sqrt{\sigma^2}}\ \exp\left(-\frac{1}{2}(x-\mu)(\sigma^2)^{-1}(x-\mu)\right)$$

Properties

① $X \sim N(\mu_x, \sigma_x^2)$

$Y \sim N(\mu_y, \sigma_y^2)$

② $Cor(X, Y) = \rho$

③ $aX + bY$

$\sim N(\underline{a\mu_x + b\mu_y}, \underline{a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\sigma_x\sigma_y\rho})$

④ $(X, Y) \sim \text{BivN} \Leftrightarrow \forall a, b \quad aX + bY \sim N$

5 Prev: If $X \perp Y$ then $Cor(X, Y) = 0$

If $(X, Y) \sim BivN$ and $\rho = 0$

then $X \perp Y$.