Lecture 20

Ex.

$$X \le Y \begin{cases} Y \sim \text{Pois}(\lambda) \\ X \mid Y = y \sim \text{Bin}(y, p) \end{cases}$$

Bin(n, p)
PMF:
$$\binom{n}{x} p^x (1-p)^{n-x}$$

What is dist of X?

$$f(x) = \sum_y \underbrace{f(x|y)}_{\text{Bin}(y,p)} \underbrace{f(y)}_{\text{Pois}(\lambda)}$$

$$= \sum_{y=x}^{\infty} \boxed{\binom{y}{x}} p^x (1-p)^{y-x} \frac{\lambda^y e^{-\lambda}}{y!}$$

$$\frac{\cancel{y!}}{x!(y-x)!} \frac{1}{\cancel{y!}} = \frac{1}{x!(y-x)!}$$

$$= \frac{\lambda^x p^x e^{-\lambda}}{x!} \sum_{y=x}^{\infty} \frac{(1-p)^{y-x} \lambda^{y-x}}{(y-x)!}$$

$$= \frac{\lambda^x p^x e^{-\lambda}}{x!} \underbrace{\sum_{y=0}^{\infty} \frac{[(1-p)\lambda]^y}{y!}}_{\exp((1-p)\lambda)}$$

$$e^{(\cdots)}$$

$$= \frac{(p\lambda)^x e^{-\lambda}}{x!} \exp((1-p)\lambda)$$

$$\boxed{f(x) = \frac{(p\lambda)^x e^{-p\lambda}}{x!}} \longrightarrow X \sim \text{Pois}(p\lambda)$$

Theorem: Iterated Expectation

If X and Y are RVs then

$$E[X] = E_Y[\underbrace{E[X|Y]}_{\text{a RV (fn of } Y)}]$$

$$E[X|Y=y] = \int_{\mathbb{R}} x f(x|y)\,dx = \text{a number}$$

For each $y \in \mathbb{R}$ we have some corresp. value

$$g(y) = E[X|Y=y]$$

e.g. $g(y) = y^2$ or $g(y) = y+1$

Can plug Y into g

e.g. $g(Y) = Y^2$ or $g(Y) = Y+1$

$$g(Y) = \cancel{E[X \mid Y = Y]}$$

weird notation

$$= E[X \mid Y]$$

a RV

punchline:

$E[X \mid Y = y]$ = a number

$E[X \mid Y]$ = a RV (got by promoting y to Y)

Ex: Prev.

$$Y \sim Pois(\lambda)$$
$$X \mid Y=y \sim Bin(y, p)$$

What's $E[X]$?

① $E[X \mid Y=y] = yp$

② $E[X \mid Y] = Yp$

③ $E[X] = E[E[X \mid Y]]$
$= E[Yp]$
$= p\,E[Y]$
$= p\lambda$

Ex. $P \sim Beta(\alpha, \beta)$

$X \mid P=p = Bin(n, p)$

$E[X]$?

① $E[X \mid P=p] = np$

② $E[X \mid P] = nP$

③ $E[X] = E[E[X \mid P]]$

$$= E[nP]$$

$$= nE[P]$$

$$= n \frac{\alpha}{\alpha+\beta}.$$

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pf. (cts)

① $\boxed{f(x) = \int_{\mathbb{R}} f(x,y)\,dy}$

② $f(x|y) = \dfrac{f(x,y)}{f(y)} \Longleftrightarrow \boxed{f(x,y) = f(x|y)f(y)}$

③ $\boxed{g(y) = E[X \mid Y=y] = \int_{\mathbb{R}} x f(x|y)\,dx}$

$$E[X] = \int_{\mathbb{R}} x \underbrace{f(x)}_{1} \, dx$$

$$= \int_{\mathbb{R}} x \overbrace{\int_{\mathbb{R}} \underbrace{f(x,y)}_{2} \, dy}^{1} \, dx$$

$$= \int_{\mathbb{R}} x \overbrace{\int_{\mathbb{R}} f(x|y) f(y) \, dy}^{2} \, dx$$

$$= \int_{\mathbb{R}} \underbrace{\int_{\mathbb{R}} x f(x|y) \, dx}_{g(y)} \, f(y) \, dy$$

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$$= \int_{\mathbb{R}} g(y) f(y) \, dy$$

$$= E[g(Y)]$$

$$= E[E[X|Y]].$$

Theorem: Law of Total Variance

$$\mathrm{Var}(X) = E[\underbrace{\mathrm{Var}(X|Y)}] + \mathrm{Var}(\underbrace{E[X|Y]})$$

similarly defined

Ex. $\boxed{P \sim \mathrm{Beta}(\alpha, \beta)}$

$X | P = p \sim \mathrm{Bin}(n, p)$

$\mathrm{Var}(X)$?

1. $E[X | P = p] = np$

   $\mathrm{Var}(X | P = p) = np(1-p)$

2. $E[X | P] = nP$

   $\mathrm{Var}(X | P) = nP(1-P)$

3. $\mathrm{Var}(X) = E[\mathrm{Var}(X|P)] + \mathrm{Var}(E[X|P])$

   $= E[nP(1-P)] + \mathrm{Var}(nP)$

$$= n\left(E[P]-E[P^2]\right) + n^2\, Var(P)$$

$E[P^2] = Var(P) + E[P]^2$

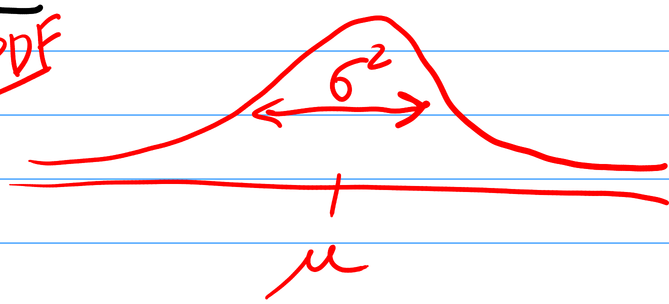
$$E[P] = \frac{\alpha}{\alpha+\beta}$$

plug in

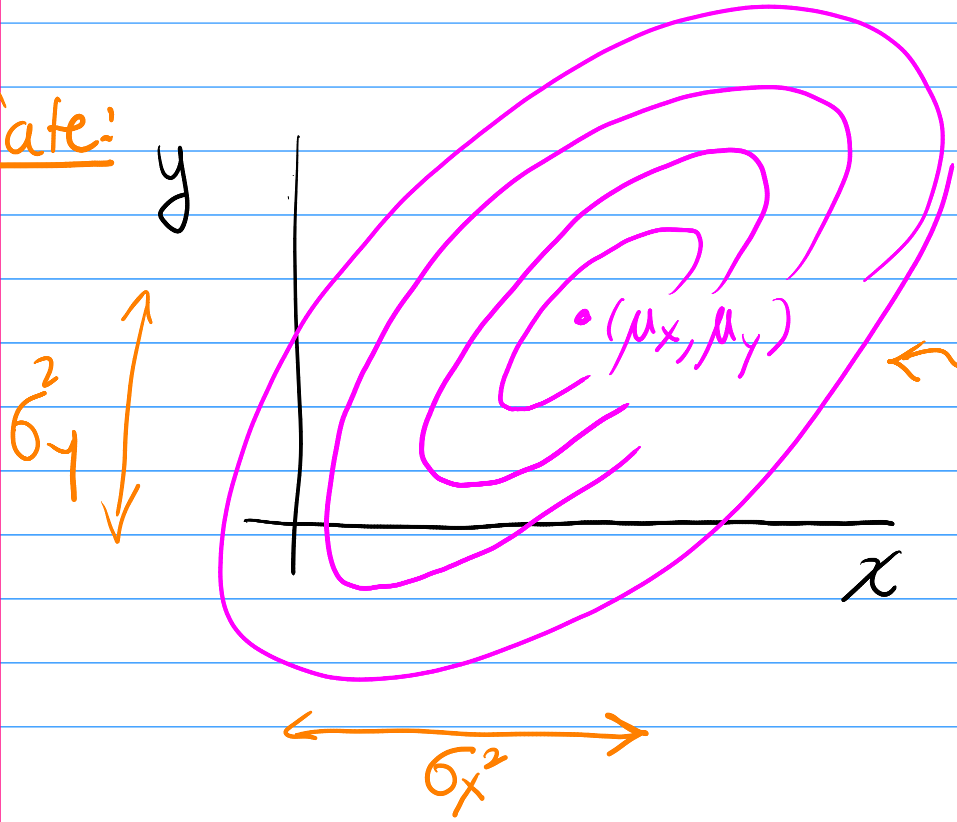
$$Var(P) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Bivariate Normal

Uni: $N(\mu, \sigma^2)$

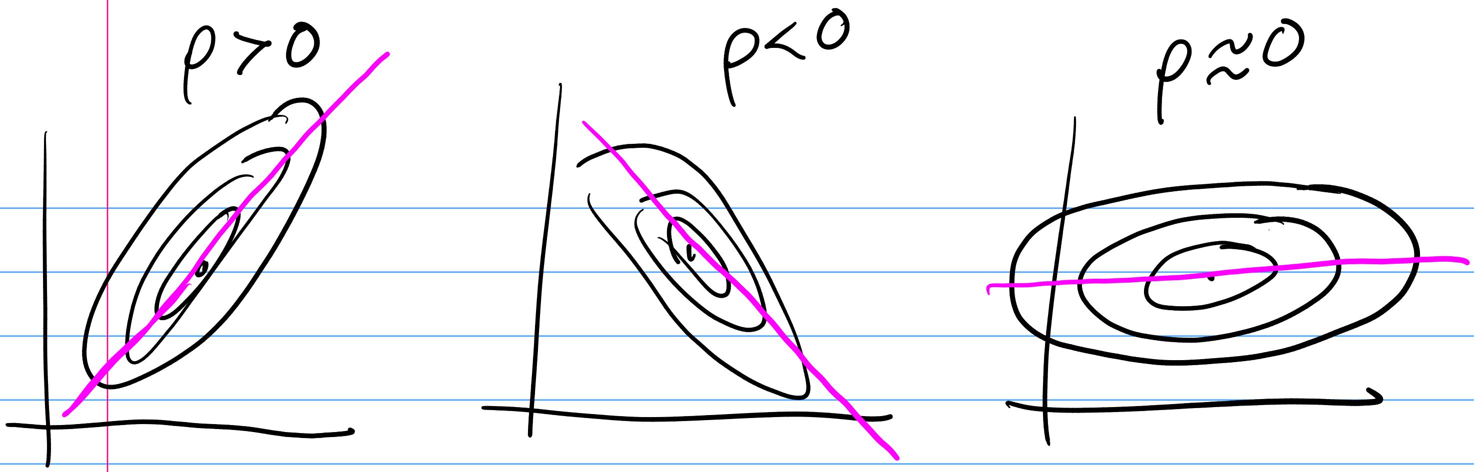


Bivariate:



how related?

ρ = correlation



$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot$$

$$\exp\left\{-\frac{1}{2\sqrt{1-\rho^2}}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]\right\}$$

<u>alt:</u> $\mu = (\mu_x, \mu_y) \in \mathbb{R}^{2\times 1}$ = mean vector

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_x\sigma_y\rho \\ \sigma_x\sigma_y\rho & \sigma_y^2 \end{bmatrix} = \begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{bmatrix}_{2\times 2}$$

Covariance matrix

$z = (x, y)$

Bivariate:

$$f(z) = \frac{1}{2\pi}\frac{1}{\sqrt{\det\Sigma}}\exp\left(-\frac{1}{2}\overbrace{\underbrace{(z-\mu)^T}_{1\times 2}\underbrace{\Sigma^{-1}}_{2\times 2}\underbrace{(z-\mu)}_{2\times 1}}^{\text{a number}}\right)$$

1×1

uni:

$$f(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\sigma^2}}\exp\left(-\frac{1}{2}(x-\mu)(\sigma^2)^{-1}(x-\mu)\right)$$

Facts:

1. $X \sim N(\mu_x, \sigma_x^2)$
   $Y \sim N(\mu_y, \sigma_y^2)$

2. $Cor(X, Y) = \rho$

3. $aX + bY \sim N(a\mu_x + b\mu_y,\ a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\rho\sigma_x\sigma_y)$

4. $(X, Y) \sim BivN \Leftrightarrow \forall a, b\ \ aX + bY \sim N(\cdots)$

5 Prev: If $X \perp Y$ then $\mathrm{Cor}(X, Y) = 0$

If $(X, Y) \sim \mathrm{BivN}$ and $\rho = 0$

then $X \perp Y$.