Lecture 21

Final 7-10 pm, May 7, ISC 1280

Transformation:

Uni: $g : \mathbb{R} \to \mathbb{R}$, what is dist $g(X)$?

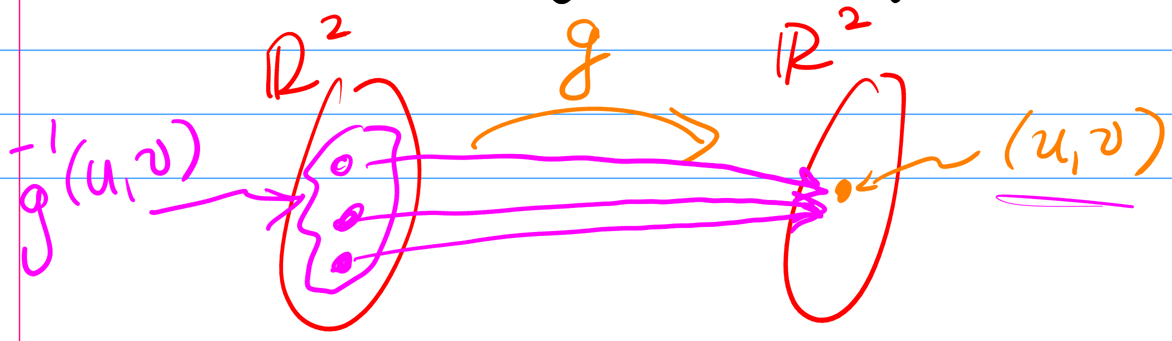
Biv: $g : \mathbb{R}^2 \to \mathbb{R}^2$ what is dist of $g(X, Y)$?

notation: $(X, Y) \xrightarrow{g} (U, V)$

e.g. $(U, V) = (X^2 Y, -\log(X))$

Discrete: Assume X and Y discrete

$$(U, V) = (g_1(X, Y), g_2(X, Y))$$



Inverse Image:

$$g^{-1}(u,v) = \{(x,y) : g(x,y) = (u,v)\}$$

Want: PMF of (U,V) from PMF (X,Y)

$$f_{U,V}(u,v) = P(U=u, V=v)$$

$$= P((U,V) \in \{(u,v)\})$$

$$= P(g(X,Y) \in \{(u,v)\})$$

$$= P((X,Y) \in \overbrace{g^{-1}(u,v)}^{\text{a set}})$$

the set of things that maps to (u,v)

$$= \sum_{(x,y) \in g^{-1}(u,v)} f_{X,Y}(x,y)$$

If g is invertible then

comps. of inverse fn.

$$= \boxed{f_{X,Y}(g_1^{-1}(u,v), g_2^{-1}(u,v))}$$

$g: \mathbb{R}^2 \to \mathbb{R}^2$

$(u, v) = g(x, y)$

$u = g_1(x, y), \quad v = g_2(x, y)$

$g^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$

$(x, y) = g^{-1}(u, v)$

$x = g_1^{-1}(u, v), \quad y = g_2^{-1}(u, v)$

Ex. Let $X \perp Y$ and

$X \sim Pois(\theta)$

$Y \sim Pois(\lambda)$

what is the dist of

$U = X + Y$

$V = Y$

$u = g_1(x, y) = x + y$ and $v = g_2(x, y) = y$

$\boxed{u - v = x + y - y = x = g_1^{-1}(u, v)}$ $\quad$ $\boxed{y = v = g_2^{-1}(u, v)}$

$$f_{U,V}(u,v) = f_{X,Y}(g_1^{-1}(u,v), g_2^{-1}(u,v))$$

Since $X \perp Y$ then

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$= \frac{\theta^x e^{-\theta}}{x!} \frac{\lambda^y e^{-\lambda}}{y!}$$

$$= f_{X,Y}(u-v, v)$$

$$\boxed{= \frac{\theta^{u-v} e^{-\theta}}{(u-v)!} \frac{\lambda^v e^{-\lambda}}{v!} \quad \text{for } \boxed{u \geq v}}$$

What's dist of $U = X+Y$?

$$f_U(u) = \sum_v f(u,v)$$

$$= \frac{1}{u!} \sum_{v=0}^{u} \frac{\theta^{u-v} e^{-\theta}}{(u-v)!} \frac{\lambda^v e^{-\lambda}}{v!} u! \quad \binom{u}{v}$$

$$= \frac{e^{-(\theta+\lambda)}}{u!} \underbrace{\sum_{v=0}^{u} \binom{u}{v} \theta^{u-v} \lambda^{v}}_{(\theta+\lambda)^u}$$

Bin Thrm

$$\boxed{(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}}$$

$$f(u) = \frac{(\theta+\lambda)^u e^{-(\theta+\lambda)}}{u!}$$

Pois$(\theta+\lambda)$

Theorem: If $X \perp Y$ and $X \sim \text{Pois}(\theta)$, $Y \sim \text{Pois}(\lambda)$

then $X+Y \sim \text{Pois}(\theta+\lambda)$.

What about cts?

Uni: If g is nice (invertible, inverse, diff'able) then if $Y = g(X)$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

Bivariate: If

- X and Y cts
- $(U,V) = (g_1(X,Y), g_2(X,Y))$
- g is invertible
- g^{-1} is diff'able

then

$$\boxed{f_{U,V}(u,v) = f_{X,Y}\left(g_1^{-1}(u,v), g_2^{-1}(u,v)\right) \left|\det J\right|}$$

J = jacobian of g^{-1}

Jacobian $\quad h: \mathbb{R}^2 \to \mathbb{R}^2$

$$h(x,y) = (h_1(x,y), h_2(x,y))$$

then the Jacobian of h is

$$J = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{bmatrix}$$

In our case we need Jacobian of g^{-1}

$$J = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix}$$

For a 2x2 mtx

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - cb$$

Ex. $(U,V) = (X+Y, X-Y)$

$$u = x+y = g_1(x,y)$$

$$v = x-y = g_2(x,y)$$

what's the density of (U,V).

(1) get inverses

$$u+v = x+y+x-y = 2x$$

$$\boxed{x = \frac{u+v}{2} = g_1^{-1}(u,v)}$$

$$u-v = x+y-(x-y) = 2y$$

$$\boxed{y = \frac{u-v}{2} = g_2^{-1}(u,v)}$$

② Get J and $|\det J|$

$$J = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\det J = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$= -\frac{1}{2}$$

$$|\det J| = \frac{1}{2}$$

③ plug into formula:

$$f(u,v) = f_{X,Y}\left(g_1^{-1}(u,v), g_2^{-1}(u,v)\right) |\det J|$$

$$\boxed{= f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \frac{1}{2}}$$

Assume $X, Y \overset{iid}{\sim} N(0,1)$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad (\text{by } \perp)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

$$f(u,v) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u+v}{2}\right)^2} \cdot e^{-\frac{1}{2}\left(\frac{u-v}{2}\right)^2} \cdot \frac{1}{2}$$

$$\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2$$

$$= \frac{u^2+v^2\cancel{+2uv}+u^2+v^2\cancel{-2uv}}{4}$$

$$= \frac{u^2+v^2}{2}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \frac{1}{2} e^{-\frac{1}{2}\left(\frac{u^2+v^2}{2}\right)}$$

$$f(u,v) = \underbrace{\frac{1}{\sqrt{2 \cdot 2\pi}} e^{-\frac{1}{2}\frac{1}{2}u^2}}_{\text{fn of } U} \cdot \underbrace{\frac{1}{\sqrt{2 \cdot 2\pi}} e^{-\frac{1}{2}\frac{1}{2}v^2}}_{\text{fn } V}$$

$N(0,2)$ $N(0,2)$

i.e. $U, V \overset{iid}{\sim} N(0,2)$