Lecture 21

Final: May 7, 7-10pm, ISC 1280 ← Weds.

Transformations:

Uni: $g: \mathbb{R} \to \mathbb{R}$, what is the dist of $g(X)$

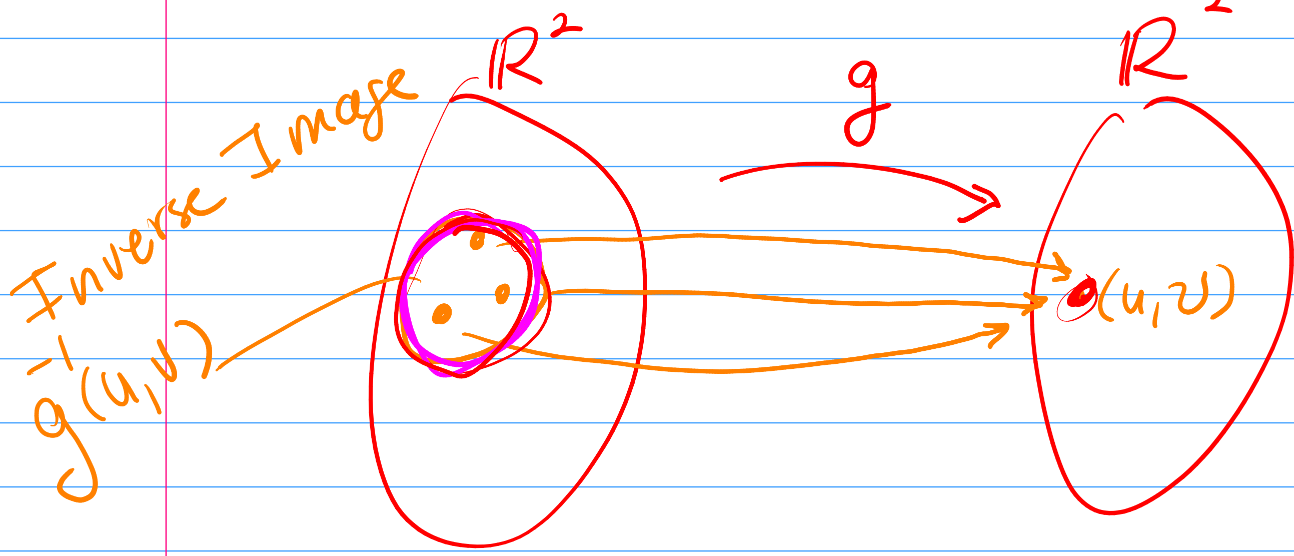
Biv: $g: \mathbb{R}^2 \to \mathbb{R}^2$ what is the dist of $g(X, Y)$

notation: $(U, V) = g(X, Y)$

$= (g_1(X, Y), g_2(X, Y))$

e.g. $(U, V) = (X^2 Y, -\log(X))$

Discrete: Assume X, Y discrete



$$g^{-1}(u,v) = \{(x,y) : g(x,y) = (u,v)\}$$

Want: Get PMF of (U,V) from PMF of (X,Y)

$$
\begin{aligned}
f_{U,V}(u,v) &= P(U=u, V=v) \\
&= P((U,V) \in \{(u,v)\}) \\
&= P(g(X,Y) \in \{(u,v)\}) \\
&= P((X,Y) \in \underbrace{g^{-1}(u,v)}_{\text{Some set}}) \\
&= \sum_{(x,y) \in g^{-1}(u,v)} f_{X,Y}(x,y)
\end{aligned}
$$

If g is invertible then

$$\boxed{f_{U,V}(u,v) = f_{X,Y}\big(\underbrace{g_1^{-1}(u,v)}, \underbrace{g_2^{-1}(u,v)}\big)}$$

component functions of g^{-1}

Ex. let $X \sim Pois(\theta)$, $Y \sim Pois(\lambda)$ $\Big\}$ $X \perp Y$

What is the dist of

$$\left.\begin{aligned} U &= X + Y \\ V &= Y \end{aligned}\right\} \quad V \leq U$$

① Find inverse:

$$\left.\begin{aligned} u &= g_1(x,y) = x+y \\ &\boxed{v = g_2(x,y) = y} \end{aligned}\right\} \quad \underline{u-v} = \underline{x+y-y} \quad \underline{= x}$$

$$\boxed{x = g_1^{-1}(u,v) = u - v}$$

$$\boxed{y = g_2^{-1}(u,v) = v}$$

② get $f_{X,Y}$

b/c independent

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

$$= \frac{\theta^x e^{-\theta}}{x!} \frac{\lambda^y e^{-\lambda}}{y!}$$

③ plug in fmla

$$f_{U,V}(u,v) = f_{X,Y}(u-v, v)$$

$$\boxed{= \frac{\theta^{u-v} e^{-\theta}}{(u-v)!} \frac{\lambda^v e^{-\lambda}}{v!} \quad \text{for } u \geq v}$$

Let's get marginal of $U = X + Y$

$$f_U(u) = \sum_v f(u,v)$$

$$= \frac{1}{u!}\sum_{v=0}^{u} \frac{\theta^{u-v} e^{-\theta}}{(u-v)!} \frac{\lambda^{v} e^{-\lambda}}{v!} u!$$

$$= \frac{e^{-(\theta+\lambda)}}{u!} \underbrace{\sum_{v=0}^{u} \binom{u}{v} \lambda^{v} \theta^{u-v}}_{(\lambda+\theta)^u}$$

Binomial Theorem

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$\underbrace{f(u) = \frac{(\lambda+\theta)^u e^{-(\lambda+\theta)}}{u!}}_{U \sim Pois(\theta+\lambda)}$$

<u>Theorem:</u> If $X \sim Pois(\theta)$

$Y \sim Pois(\lambda)$

and $X \perp Y$

then $X + Y \sim Pois(\theta + \lambda)$

What about cts RVs?

Uni: If g is invertible, g^{-1} diff'able then $Y = g(X)$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

Bivariate:

- X and Y cts
- $(U, V) = (g_1(X, Y), g_2(X, Y))$
- g is invertible
- g^{-1} is diff'able

$$f_{U,V}(u,v) = f_{X,Y}(g_1^{-1}(u,v), g_2^{-1}(u,v)) \left| \det J \right|$$

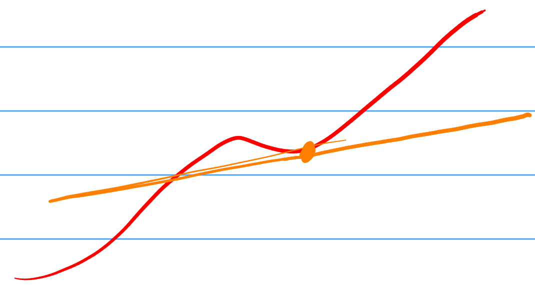
J = jacobian of g^{-1}

Jacobians

$$h : \mathbb{R}^2 \to \mathbb{R}^2$$

$$h(x,y) = (h_1(x,y),\ h_2(x,y))$$

the Jacobian of h is

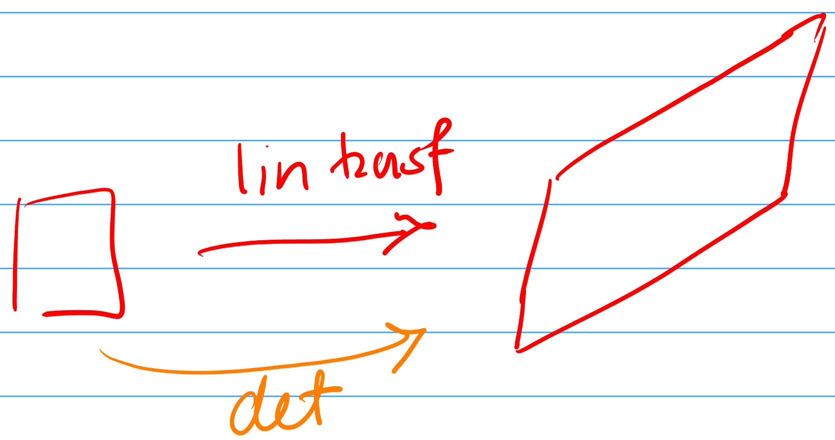


$$J = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{bmatrix}$$

In our case:

$$J = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix}$$

Determinant:



For a 2x2 mtx

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det A = ad - cb$$

Ex, $X, Y \overset{iid}{\sim} N(0,1)$

$(U, V) = (X+Y, X-Y)$

what's dist of U, V?

1 get inverse functions

$u = g_1(x,y) = x+y$

$v = g_2(x,y) = x-y$

Solve for x, y in terms of u, v

$u+v = x+y+x-y = 2x$

$$\boxed{x = \frac{u+v}{2} = g_1^{-1}(u,v)}$$

$$u - v = x + y - (x - y) = 2y$$

$$\boxed{y = \frac{u - v}{2} = g_2^{-1}(u, v)}$$

② Get J and $\det J$

$$J = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\det J = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}$$

$$|\det J| = 1/2 .$$

③ Get $f_{X,Y}$

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

4 Plug in

$$f_{U,V}(u,v) = f_{X,Y}\left(\tfrac{1}{2}(u+v), \tfrac{1}{2}(u-v)\right)\cdot \frac{1}{2}$$

$$\boxed{= \frac{1}{2}\frac{1}{2\pi} e^{-\frac{1}{2}\left(\frac{1}{2}(u+v)\right)^2} e^{-\frac{1}{2}\left(\frac{1}{2}(u-v)\right)^2}}$$

$$\left[\tfrac{1}{2}(u+v)\right]^2 + \left[\tfrac{1}{2}(u-v)\right]^2$$

$$= \frac{1}{4}\left(u^2+v^2\cancel{+2uv}+u^2+v^2\cancel{-2uv}\right)$$

$$= \frac{1}{4}(2u^2+2v^2) = \frac{1}{2}(u^2+v^2)$$

$$= \frac{1}{2}\frac{1}{2\pi}\exp\left(-\frac{1}{2}\frac{1}{2}(u^2+v^2)\right)$$

$$= \underbrace{\frac{1}{2}\frac{1}{2\pi} e^{-\frac{1}{2}\frac{1}{2}u^2}}_{\text{fn } u} \cdot \underbrace{e^{-\frac{1}{2}\frac{1}{2}v^2}}_{\text{fn } v} = f(u,v)$$

so $U \perp\!\!\!\perp V$

$$= \underbrace{\frac{1}{\sqrt{2\cdot 2\pi}} e^{-\frac{1}{2}\frac{1}{2}u^2}}_{N(0,2)} \cdot \underbrace{\frac{1}{\sqrt{2\cdot 2\pi}} e^{-\frac{1}{2}\frac{1}{2}v^2}}_{N(0,2)}$$