Lecture 22

Ex. $X \sim \text{Gamma}(\alpha, \lambda)$, $Y \sim \text{Gamma}(\beta, \lambda)$, $X \perp Y$

$U = X + Y$ and $V = \dfrac{X}{X+Y}$

what's the dist of U and V?

$$f(u,v) = f_{X,Y}\left(g_1^{-1}(u,v),\, g_2^{-1}(u,v)\right)|\det J|$$

① Get Inverses

$$u = g_1(x,y) = x+y$$

$$v = g_2(x,y) = x/x+y$$

$$uv = (x+y)\left(\frac{x}{x+y}\right) = x$$

So $\boxed{x = g_1^{-1}(u,v) = uv}$

$$u - uv = x + y - x = y$$

so $\boxed{y = g_2^{-1}(u,v) = u(1-v)}$

(2) Get $f_{X,Y}$

Since $X \perp Y$ then

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

$$\boxed{= \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \frac{\lambda e^{-\lambda y}(\lambda y)^{\beta-1}}{\Gamma(\beta)}}$$

(3) Get $|\det J|$

$$J = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix} = \begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix}$$

$$\det J = (v)(-u) - (u)(1-v)$$

$$= -uv - u + uv = -u$$

$$\boxed{|\det J| = |-u| = u}$$

④ Plug-in

$$f(u,v) = f_{X,Y}(uv,\ u(1-v))\ u$$

$$\boxed{= \frac{\lambda e^{-\lambda uv}(\lambda uv)^{\alpha-1}}{\Gamma(\alpha)} \frac{\lambda e^{-\lambda u(1-v)}(\lambda u(1-v))^{\beta-1}}{\Gamma(\beta)}\, u \quad \text{for } u>0 \text{ and } 0<v<1}$$

$= \cdots$ algebra

$$f(u,v) = \underbrace{\frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)}}_{\text{const}} \underbrace{u^{\alpha+\beta-1} e^{-\lambda u}}_{\text{only } u} \underbrace{v^{\alpha-1}(1-v)^{\beta-1}}_{\text{only } v}$$

$U \perp V$

claim: $\boxed{U \sim \text{Gamma}(\alpha+\beta, \lambda)}$

$V \sim \text{Beta}(\alpha, \beta)$

Theorem: Transf. and Independence

If $X \perp Y$ and

$$U = g(X) \leftarrow \text{no } Y$$

$$V = h(Y) \leftarrow \text{no } X$$

then $U \perp V$.

Ex. $U = X^2$ and $V = \log Y$ ...

Ex. $U = XY$ and $V = X$ — assume $X, Y > 0$

What's the dist of U and V?

① Get Inverses

$$u = g_1(x,y) = xy$$

$$v = g_2(x,y) = x \rightarrow \boxed{x = v = g_1^{-1}(u,v)}$$

$$\frac{u}{v} = \frac{xy}{x} = y$$

so $\boxed{y = g_2^{-1}(u,v) = u/v}$

② Get det J

$$J = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/v & -u/v^2 \end{bmatrix}$$

$$\det J = (0)\left(\frac{-u}{v^2}\right) - \left(\frac{1}{v}\right)(1)$$

$$= -1/v$$

$$|\det J| = 1/v$$

③ Plug-in

$$\boxed{f(u,v) = f_{X,Y}\left(v, \frac{u}{v}\right)\frac{1}{v}}$$

Multivariate RVs

If $X_1, \dots, X_n$ are RVs then

$$\underset{\sim}{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

is called a multivariate RV or a random vector.

Defn: PMF/PDF

If the X_i are discrete then the joint PMF is the function

$$\underset{\sim}{x} \in \mathbb{R}^n$$

$$f(\underset{\sim}{x}) = f(x_1, x_2, \ldots, x_n)$$

$$= P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

If the X_i's are cts then the joint PDF is the function $f: \mathbb{R}^n \to \mathbb{R}$ so that for any $A \subset \mathbb{R}^n$

$$P(\underset{\sim}{X} \in A) = \int_A f(\underset{\sim}{x})\, d\underset{\sim}{x}$$

$$= \int_A \cdots \int f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

Expectation: If $g: \mathbb{R}^n \to \mathbb{R}$ then

$$E[g(\underset{\sim}{X})] = \begin{cases} \sum_{x_1} \sum_{x_2} \cdots \sum_{x_n} g(x_1, \ldots, x_n) f(x_1, \ldots, x_n) & \text{discrete} \\ \int_{\mathbb{R}^n} \cdots \int g(x_1, \ldots, x_n) f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n & \end{cases}$$

Defn: Marginal Dist

Discrete:

skipped i

$$f(x_i) = \sum_{x_1} \sum_{x_2} \cdots \sum_{x_{i-1}} \sum_{x_{i+1}} \cdots \sum_{x_n} f(x_1, \dots, x_n)$$

Cts:

$$f(x_i) = \int \cdots \int f(x_1, \dots, x_n)\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_n$$

We can get the marginal dist of any seq

$$X_{i_1}, X_{i_2}, \dots, X_{i_m}$$

by summing/integrating out the other vars.

Ex.

$$f(x_2, x_4) = \int \cdots \int f(x_1, \dots, x_n)\, dx_1\, dx_3\, dx_5 \cdots dx_n$$

Defn: Conditional Dist

If I have two sets of vars:

$$\mathbb{X}_1, \ldots, \mathbb{X}_n \text{ and } \mathbb{Y}_1, \ldots, \mathbb{Y}_m$$

the conditional of the $\mathbb{X}$s given the $\mathbb{Y}$s is:

$$f(\underset{\sim}{x} \mid \underset{\sim}{y}) = f(x_1, \ldots, x_n \mid y_1, \ldots, y_m)$$

$$= \frac{f(x_1, \ldots, x_n, y_1, \ldots, y_m)}{f(y_1, \ldots, y_m)}$$

Ex. $\mathbb{X}_1, \ldots, \mathbb{X}_4$ have the dist (PDF)

$$f(x_1, \ldots, x_4) = \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)$$

for $0 < x_i < 1$.

(a) $P(X_1 < \frac{1}{2}, X_2 < \frac{3}{4}, X_4 > 1/2)$

$$= \int_{1/2}^{1}\int_{0}^{1}\int_{0}^{3/4}\int_{0}^{1/2} \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)\,dx_1\,dx_2\,dx_3\,dx_4$$

$= \dots$

$= 3/256$

(b) what's the dist of X_1 and X_2?

$$f(x_1, x_2) = \iint f(x_1, \dots, x_4)\,dx_3\,dx_4$$

$$= \int_0^1\int_0^1 \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)\,dx_3\,dx_4$$

$= \dots$

$$= \underbrace{\frac{1}{2} + \frac{3}{4}(x_1^2 + x_2^2)}$$

(c)

$$E[X_1 X_2] \quad g(x_1,\dots,x_4)$$

(1) $$= \int \cdots \int \overbrace{x_1 x_2}\, f(x_1,\dots,x_4)\, dx_1 \cdots dx_4$$

$$= \int_0^1\int_0^1\int_0^1\int_0^1 x_1 x_2 \, \frac{3}{4}(x_1^2 + x_2^2 + x_3 + x_4^2)\, dx_1 \cdots dx_4$$

(2) $$= \iint x_1 x_2 f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_0^1\int_0^1 x_1 x_2 \left(\frac{1}{2} + \frac{3}{4}(x_1^2 + x_2^2)\right) dx_1\, dx_2$$

$$= \cdots = 5/16$$

(d) Conditional Dist

$$f(x_3, x_4 \mid x_1, x_2) = \frac{f(x_1, \ldots, x_4)}{f(x_1, x_2)}$$

$$= \frac{\frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)}{\frac{1}{2} + \frac{3}{4}(x_1^2 + x_2^2)}$$