Lecture 22

Ex.

$$X \sim \text{Gamma}(\alpha, \lambda)$$
$$Y \sim \text{Gamma}(\beta, \lambda)$$

$X \perp Y$

$$U = X + Y \quad \text{and} \quad V = \frac{X}{X+Y}$$

What's the dist of U and V?

$$f(u,v) = f_{X,Y}\left(g_1^{-1}(u,v), g_2^{-1}(u,v)\right) \left|\det J\right|$$

1 Get inverses

$$u = g_1(x,y) = x + y$$
$$v = g_2(x,y) = x/x+y$$

$$uv = x+y \frac{x}{x+y} = x$$

$$\boxed{x = g_1^{-1}(u,v) = uv}$$

$$u - uv = x + y - x = y$$

$$\boxed{y = g_2^{-1}(u,v) = u(1-v)}$$

② Get $|\det J|$

$$J = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix} = \begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix}$$

$$\det J = (v)(-u) - (1-v)(u)$$

$$= -uv - u + uv$$

$$= -u$$

$$|\det J| = |-u| = |u| = u$$

③ Get $f_{X,Y}$

b/c $X \perp Y$ then

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$= \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \frac{\lambda e^{-\lambda y}(\lambda y)^{\beta-1}}{\Gamma(\beta)}$$

④ Plug-in

$$f(u,v) = \underbrace{f_{X,Y}(uv, u(1-v))\, u}$$

$$\boxed{= \frac{\lambda e^{-\lambda uv}(\lambda uv)^{\alpha-1}}{\Gamma(\alpha)} \frac{\lambda e^{-\lambda u(1-v)}(\lambda u(1-v))^{\beta-1}}{\Gamma(\beta)}\, u \quad \text{for } u>0,\ 0<v<1}$$

= algebra ...

$$f(u,v) = \underbrace{\frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)}}_{\text{const.}} \underbrace{\boxed{u^{\alpha+\beta-1} e^{-\lambda u}}}_{\text{fn of } u} \underbrace{\boxed{v^{\alpha-1}(1-v)^{\beta-1}}}_{\text{fn of } v}$$

So $U \perp V$

$U \sim \text{Gamma}(\alpha+\beta, \lambda)$

$V \sim \text{Beta}(\alpha, \beta)$

Theorem: Transf. and Independence

Let $X \perp Y$ and

$U = g(X)$ no Y

$V = h(Y)$ no X

then $U \perp V$.

Ex. $U = X^2$ and $V = \log(Y)$.

$U \perp V$ if $X \perp Y$.

Ex. Assume $X, Y > 0$

$U = XY$ and $V = X$.

what's the dist of U, V?

① Get Inverse

$u = g_1(x,y) = xy$

$v = g_2(x,y) = x$

$\frac{u}{v} = \frac{xy}{x} = y$

$$\boxed{\begin{aligned} x &= g_1^{-1}(u,v) = v \\ y &= g_2^{-1}(u,v) = u/v \end{aligned}}$$

② Get J and |det J|

$$J = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/v & -u/v^2 \end{bmatrix}$$

$$\det J = (0)\left(-u/v^2\right) - \left(\frac{1}{v}\right)(1)$$

$$= -1/v$$

$$\boxed{|\det J| = \left|\frac{1}{v}\right| = 1/v}$$

③ Plug-in

$$\boxed{f(u,v) = f_{X,Y}(v, u/v)\frac{1}{v}}$$

Multivariate RVs

If $X_1, \dots, X_n$ are RVs then

$$\underset{\sim}{X} = \begin{bmatrix} X_1 \\ \vdots \\ \vdots \\ X_n \end{bmatrix}$$

is called a <u>multivariate RV</u>

or a <u>random vector</u>.

<u>Defn: PMF/PDF</u>

If the X_i are discrete then the joint PMF of the Xs is

$\underset{\sim}{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$

$$f(\underset{\sim}{x}) = f(x_1, \dots, x_n)$$

$$= P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$$

If the X_i are cts then the joint PDF is function $f: \mathbb{R}^n \to \mathbb{R}$ so that for $A \subset \mathbb{R}^n$ then

$$P(\underset{\sim}{X} \in A) = \int_A f(\underset{\sim}{x})\, d\underset{\sim}{x}$$

$$= \int \cdots \int_A f(x_1, \dots, x_n)\, dx_1\, dx_2 \cdots dx_n .$$

Expectation:

If $g: \mathbb{R}^n \to \mathbb{R}$ then

$$E[\underbrace{g(\underset{\sim}{X})}_{\mathbb{R}}] = \begin{cases} \sum_{x_1}\sum_{x_2}\sum_{x_3}\cdots\sum_{x_n} g(x_1,\dots,x_n) f(x_1,\dots,x_n) & \text{discrete} \\ \int \cdots \int_{\mathbb{R}^n} g(x_1,\dots,x_n) f(x_1,\dots,x_n)\, dx_1 \cdots dx_n \end{cases}$$

Defn: Marginal Dists

Discrete:

$$\boxed{f(x) = \sum_y f(x,y)}$$

$$f(x_i) = \sum_{x_1}\sum_{x_2}\cdots\sum_{x_{i-1}}\sum_{x_{i+1}}\cdots\sum_{x_n} f(x_1,\dots,x_n)$$

Cts:

$$f(x_i) = \int\cdots\int f(x_1,\dots,x_n)\, dx_1 \cdots d_{x_{i-1}}\, d_{x_{i+1}} \cdots dx_n$$

In general:

We can get the dist of some seq

$$X_{i_1}, \dots, X_{i_m}$$

by summing / integrating out all other vars.

Ex.

$$f(x_3, x_7) = \int \cdots \int f(x_1, \dots, x_n) \underbrace{dx_1 dx_2 dx_4 dx_5 dx_6 dx_8 \cdots dx_n}_{\text{all but } X_3, X_7}$$

Conditional Dists

If I have two sets of vars

$$X_1, \dots, X_n \text{ and } Y_1, \dots, Y_m$$

the cond. dist of the Xs given Ys is:

$$f(x_1, \dots, x_n \mid y_1, \dots, y_m) = \frac{f(x_1, \dots, x_n, y_1, \dots, y_m)}{f(y_1, \dots, y_m)}$$

Ex: Let $X_1, \dots, X_4$ w/ PDF

$$f(x_1, \dots, x_4) = \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)$$

for $0 < x_i < 1$.

(a) $P(X_1 < \frac{1}{2}, X_2 < \frac{3}{4}, X_4 > \frac{1}{2})$

$$= \int_A f(\underline{x})\, d\underline{x}$$

$$= \int_{1/2}^{1}\int_0^1\int_0^{3/4}\int_0^{1/2} \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)\, dx_1\, dx_2\, dx_3\, dx_4$$

$= \dots$

$= 3/256$

(b) What's the dist of X_1 and X_2?

$$f(x_1, x_2) = \iint_{\mathbb{R}^2} f(\underline{x})\, dx_3\, dx_4$$

$$= \int_0^1\int_0^1 \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)\, dx_3\, dx_4$$

$$= \cdots$$

$$= \frac{1}{2} + \frac{3}{4}(x_1^2 + x_2^2)$$

(c)

$E[X_1 X_2]$

(1) $$= \int_0^1\int_0^1 x_1 x_2 \underbrace{f(x_1, x_2)}\, dx_1\, dx_2$$

(2) $$= \int_0^1\int_0^1\int_0^1\int_0^1 x_1 x_2 f(x_1, \ldots, x_4)\, dx_1 \cdots dx_4$$

$\cdots = 5/16$

(d) Conditional Dist

$$f(x_3, x_4 \mid x_1, x_2) = \frac{f(x_1, x_2, x_3, x_4)}{f(x_1, x_2)}$$

$$= \frac{\frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)}{\frac{1}{2} + \frac{3}{4}(x_1^2 + x_2^2)}$$