Lecture 23

Mutual Independence of RVs

We say that $X_1, \dots, X_n$ are mutually indep. if for any sets $A_1, \dots, A_n \subset \mathbb{R}$ we have that

$$P(X_1 \in A_1, X_2 \in A_2, \dots, X_n \in A_n)$$

$$=$$

$$P(X_1 \in A_1) P(X_2 \in A_2) \cdots P(X_n \in A_n)$$

Theorem: Factorization

If the support of $\underset{\sim}{X}$ is a product space then the following 3 statements are equivalent:

1. $X_1, \dots, X_n$ are indep.
2. $f(x_1, \dots, x_n) = f(x_1) \cdots f(x_n)$
3. $F(x_1, \dots, x_n) = F(x_1) \cdots F(x_n)$

Theorem:

If $X_1, \ldots, X_n$ are indep. then

(1) If $g_i : \mathbb{R} \to \mathbb{R}$ then

$g_1(X_1), g_2(X_2), \ldots, g_n(X_n)$

are independent.

(2) $E[\underbrace{X_1 X_2 \cdots X_n}_{\text{multiply}}] = E[X_1]E[X_2]\cdots E[X_n]$

Corollary: If X_i are indep. and

$$Z = \sum_{i=1}^{n} X_i$$

then

$$M_Z(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

more generally, if

$$Z = \sum_{i=1}^{n} (a_i X_i + b_i)$$

then

$$M_Z(t) = e^{t\sum_{i=1}^{n} b_i} \prod_{i=1}^{n} M_{X_i}(a_i t)$$

Ex. If $X_i \overset{\text{indep}}{\sim} N(\mu_i, \sigma_i^2)$

and independent.

$$Y = \sum_{i=1}^{n} (a_i X_i + b_i)$$

$$\sim N\left(\sum_{i=1}^{n}(a_i\mu_i + b_i), \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$$

Multivariate Transformation

If $g : \mathbb{R}^n \to \mathbb{R}^n$ and let

$$U = g(\underset{\sim}{X})$$

n-component rand. vector

and if $\underset{\sim}{X}$ has cts components, and

① g is invertible

② g^{-1} is diff'able

then

$$f_U(u) = f_{\underset{\sim}{X}}(g^{-1}(u)) \, |\det J|$$

J is $n \times n$

$$J_{ij} = \frac{\partial g_i^{-1}}{\partial u_j}$$

Means / Variances for MV RVs

Uni: $E[X] \in \mathbb{R}$

$$Var(X) = E[(X - EX)^2] \geq 0$$

MV:

$$\underset{\sim}{X} = \begin{bmatrix} X_1 \\ \vdots \\ \vdots \\ X_n \end{bmatrix}$$

mean: $$\mu = E[\underset{\sim}{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix} \in \mathbb{R}^n$$

Covariance Matrix:

~~Variance~~: $\Sigma = Cov(\underset{\sim}{\mathbb{X}}) \in \mathbb{R}^{n \times n}$

$$\Sigma_{ij} = Cov(\mathbb{X}_i, \mathbb{X}_j)$$

note: $\Sigma_{ii} = Cov(\mathbb{X}_i, \mathbb{X}_i) = Var(\mathbb{X}_i)$

$$\Sigma = \begin{bmatrix} Var(\mathbb{X}_1) & Cov(\mathbb{X}_1, \mathbb{X}_2) & \cdots \\ Cov(\mathbb{X}_2, \mathbb{X}_1) & Var(\mathbb{X}_2) & \\ \vdots & & \ddots \end{bmatrix}_{n \times n}$$

$$Cov(\underset{\sim}{\mathbb{X}}) = E\Big[\overbrace{\overbrace{(\underset{\sim}{\mathbb{X}} - E[\underset{\sim}{\mathbb{X}}])}^{n \times 1}\overbrace{(\underset{\sim}{\mathbb{X}} - E[\underset{\sim}{\mathbb{X}}])^T}^{1 \times n}}^{n \times n}\Big]$$

Uni: $Var(\mathbb{X}) = E[(\mathbb{X} - E[\mathbb{X}])^2]$

Theorem:

Assume $a \in \mathbb{R}^m$, $B \in \mathbb{R}^{m \times n}$ then

(1) $E[\underbrace{a}_{m \times 1} + \underbrace{B}_{m \times n}\underbrace{\underset{\sim}{\mathbb{X}}}_{n \times 1}] = a + B\,E[\underset{\sim}{\mathbb{X}}]$

(2) $$\underbrace{\mathrm{Cov}(a+B\underset{\sim}{X})}_{m\times m} = \underbrace{\underbrace{B}_{m\times n}\,\underbrace{\mathrm{Cov}(\underset{\sim}{X})}_{n\times n}\,\underbrace{B^T}_{n\times m}}_{m\times m}$$

Multivariate Normal

$$\underset{\sim}{X} \sim N(\underset{n\text{-vector}}{\mu}, \underset{n\times n}{\Sigma})$$

$$f(\underset{\sim}{x}) = (2\pi)^{-n/2}(\det\Sigma)^{-1/2}\exp\left(-\frac{1}{2}(\underset{\sim}{x}-\mu)^T\Sigma^{-1}(\underset{\sim}{x}-\mu)\right)$$

Special Case: $\mu = 0$ and $\Sigma = I$
called the standard MV norm.

Theorem: if $\underset{\sim}{X} \sim N(\mu, \Sigma)$ and $a \in \mathbb{R}^m$, $B \in \mathbb{R}^{m\times n}$ then

$$a + B\underset{\sim}{X} \sim N(a + B\mu, B\Sigma B^T).$$

FINAL

The cov. mtx. Σ is

1. Symmetric $(\Sigma = \Sigma^T)$
2. pos (semi-) definite

A Pos semi-def: $x^T A x \geq 0$
gen. of $A \geq 0$

A pos def: $x^T A x > 0 \ \forall x \neq 0$
gen. of $A > 0$

Indicator Functions

$$\mathbb{1}(\text{statement}) = \begin{cases} 1 & \text{statement true} \\ 0 & \text{else} \end{cases}$$

$X \sim \text{Exp}(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$
$$= \lambda e^{-\lambda x} \mathbb{1}(x > 0)$$

Often: $f(x) = \sim\sim\sim \mathbb{1}(x \in \text{Support})$

Checking independence:

$$f(x,y) = \lambda e^{-\lambda x} e^{-y} \quad \text{for } x>0 \text{ and } y>0$$

$$= \lambda e^{-\lambda x} e^{-y} \mathbb{1}(x>0)\mathbb{1}(y>0)$$

$$= \underbrace{\left(\lambda e^{-\lambda x} \mathbb{1}(x>0)\right)}_{\text{fn of } x} \underbrace{\left(e^{-y} \mathbb{1}(y>0)\right)}_{\text{fn } y}$$