Lecture 23

Mutual Independence

We say $X_1, \dots, X_n$ are mutually independent if for any $A_1, \dots, A_n \subset \mathbb{R}$ we have

$$P(X_1 \in A_1, X_2 \in A_2, \dots, X_n \in A_n) = P(X_1 \in A_1) \cdots P(X_n \in A_n)$$

Theorem: Factorization

If the support of $\underset{\sim}{X}$ is a product space then the following are equivalent:

1. X_i are independent
2. $f(x_1, \dots, x_n) = f(x_1) \cdots f(x_n)$
3. $F(x_1, \dots, x_n) = F(x_1) \cdots F(x_n)$

Theorem: Assume $X_1, \dots, X_n$ are independent then

① If $g_i : \mathbb{R} \to \mathbb{R}$ then

$$g_1(X_1), g_2(X_2), \dots, g_n(X_n)$$

are independent.

② $E[X_1 X_2 \cdots X_n] = E[X_1]E[X_2] \cdots E[X_n]$

E.g. If X, Y, Z are indep then

$$E[X^2 \log(Y) e^Z] = E[X^2] E[\log Y] E[e^Z]$$

Corollary: If X_i are independent and

$$Z = \sum_{i=1}^{n} X_i$$

then

$$M_Z(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

more generally if

$$Z = \sum_{i=1}^{n} (a_i X_i + b_i)$$

then $M_Z(t) = e^{t\sum_{i=1}^{n} b_i} \prod_{i=1}^{n} M_{X_i}(a_i t)$

Ex. If $X_i \overset{\text{indep}}{\sim} N(\mu_i, \sigma_i^2)$

then

$$Y = \sum_{i=1}^{n} X_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

Multivariate Transformation

If $g: \mathbb{R}^n \to \mathbb{R}^n$ and

$$\underset{\sim}{U} = g(\underset{\sim}{X})$$

n×1 n×1

If $\underset{\sim}{X}$ has cts components and

1. g is invertible
2. g^{-1} is diff'able

then

$$f(\underset{\sim}{u}) = f_{\underset{\sim}{X}}(g^{-1}(\underset{\sim}{u}))\,|\det J|$$

$\hookrightarrow J$ is $n \times n$

$$J_{ij} = \frac{\partial g_i^{-1}}{\partial u_j}$$

Means / Variances for Rand. Vectors.

Means:

Uni: $E[X] \in \mathbb{R}$

multivariate: $\underset{\sim}{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$

$$\mu = E[\underset{\sim}{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix} \in \mathbb{R}^n$$

$X \sim f$

$E[X] = \int_{\mathbb{R}} x f(x)\,dx$

$y = g(x)$

$dy = dg$

$E[g(X)] = \int_{\mathbb{R}} g(x) f(x)\,dx$

$\parallel$

$Y = g(X)$, $E[Y] = \int_{\mathbb{R}} y f_Y(y)\,dy$

$$\underset{\sim}{U} = g(\underset{\sim}{X}) = \left(g_1(\underset{\sim}{X}), g_2(\underset{\sim}{X}), \ldots, g_m(\underset{\sim}{X})\right)$$

m   n

$$E[\underset{\sim}{U}] = \begin{bmatrix} E[g_1(\underset{\sim}{X})] \\ \vdots \\ E[g_m(\underset{\sim}{X})] \end{bmatrix}$$

m

<u>Covariance matrix</u>:

n-vector

$$\Sigma = \mathrm{Cov}(\underset{\sim}{X}) \in \mathbb{R}^{n\times n}$$

$$\Sigma_{ij} = \mathrm{Cov}(X_i, X_j)$$

<u>notice</u>: $\Sigma_{ii} = \mathrm{Cov}(X_i, X_i) = \mathrm{Var}(X_i)$

<u>Fact</u>: $\mathrm{Var}(X) = E[(X - EX)^2]$

$$\mathrm{Cov}(\underset{\sim}{X}) = E\Big[\underbrace{(\underset{\sim}{X} - E[\underset{\sim}{X}])}_{n\times 1}\underbrace{(\underset{\sim}{X} - E[\underset{\sim}{X}])^T}_{1\times n}\Big]$$

<u>Theorem:</u> If $a \in \mathbb{R}^m$, $B \in \mathbb{R}^{m \times n}$ and $\underset{\sim}{X}$ is n-component then

① $E[\underbrace{a}_{m\times 1} + \underbrace{\underbrace{B}_{m\times n}\underbrace{\underset{\sim}{X}}_{n\times 1}}_{m\times 1}] = a + B\,E[\underset{\sim}{X}]$

② $Cov(a + B\underset{\sim}{X}) = B\,Cov(\underset{\sim}{X})B^T$

Multivariate Normal

$$\underset{\sim}{X} \sim N(\underbrace{\mu}_{\mathbb{R}^n}, \underbrace{\Sigma}_{\mathbb{R}^{n\times n}})$$

<u>density:</u>

$$f(\underset{\sim}{x}) = (2\pi)^{-n/2}(\det\Sigma)^{-1/2}\exp\left(-\frac{1}{2}(\underset{\sim}{x}-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

<u>Special Case:</u> standard MV normal

$\mu = 0$ and $\Sigma = I$

Theorem:

If $\underset{\sim}{X} \sim N(\mu, \Sigma)$ and $a \in \mathbb{R}^m$, $B \in \mathbb{R}^{m \times n}$

$\underset{\sim}{X}$ ← n

then

$$\underbrace{a + B\underset{\sim}{X}}_{m} \sim N(a + B\mu, \underbrace{B\Sigma B^T}_{m \times m})$$

↑ FINAL ↑

Indicator Functions

$$\mathbb{1}(\text{statement}) = \begin{cases} 1 & \text{statement true} \\ 0 & \text{else} \end{cases}$$

$X \sim \text{Exp}(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

$$= \lambda e^{-\lambda x} \mathbb{1}(x > 0)$$

Independence :

$$f(x,y) = \lambda e^{-\lambda x} e^{-y} \quad \text{for } x>0,\ y>0$$

$$= \lambda e^{-\lambda x} e^{-y} \underbrace{\mathbb{1}(x>0)\mathbb{1}(y>0)}_{\mathbb{1}(A \text{ and } B) = \mathbb{1}(A)\mathbb{1}(B)}$$

$$= \underbrace{\lambda e^{-\lambda x} \mathbb{1}(x>0)}_{\text{fn } x} \cdot \underbrace{e^{-y} \mathbb{1}(y>0)}_{\text{fn } y}$$

Cov. mtx

For uni: $\sigma^2 > 0$

For multi-var: Σ

1. Σ is symmetric

2. Σ is positive (semi-) definite

Pos-def: if $\underbrace{x^T \Sigma x}_{\text{gen. of } \Sigma > 0} > 0 \quad \forall x \neq 0$

Pos semi-def : if $x^T \Sigma x \geq 0$

gen. $\Sigma \geq 0$.