Lecture 2: Probability

Defn: Sample Space : S

the set of possible outcomes.

Ex. Flip a coin:

$$S = \{H, T\}$$

Ex. Roll a six-sided die

$$S = \{1, 2, 3, \ldots, 6\}$$

Ex. Roll two dice:

$$S = \{(1,1), (1,2), (2,1), \ldots, (6,6)\}$$

Ex. Waiting time for a bus

$$S = [0, \infty)$$

Ex. Number of customers arriving

$$S = \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$$

Types of sample spaces:

1. finite: $|S| < \infty$
2. infinite:
   - (i) countable
   - (ii) uncountable

Defn: Outcomes

We call $s \in S$ an "outcome"

outcome

sample space

Defn: Events

An event is a subset of S

$$E \subset S$$

event

sample space

Ex. $S = \{1, \ldots, 6\}$

$E = \{1, 2\} \subset S$

event I roll a 1 or a 2.

Ex. $S = \{(i,j),\ 1 \le i \le 6,\ 1 \le j \le 6\}$

$E = \{(1,2), (3,2)\} \subset S$

$F = \{(1,2), (2,3)\} \subset S$

We say an event E "happens" if the outcome of our experiment is in E.

Ex. $S \subset S$

So S is an event,

its the event that something happens

Ex. $\emptyset \subset S$

So $\emptyset$ is an event

Axiomatic Probability

Given a sample space S

Want: For any event E want to assign some measure of the prob. of E happening.

Mathy: For each $E \subset S$ we assign $P(E)$

prob. of E occurring

What are the rules for P?

1. mathematically consistent
2. encode some intuition about prob.

Defn: Prob. Function P

Given a sample space S a prob. function P

is a function

$$P : 2^S \to \mathbb{R}$$

that satisfies the Kolmogorov Axioms

1. non-negative:

$$P(E) \geq 0 \quad \forall E \subset S$$

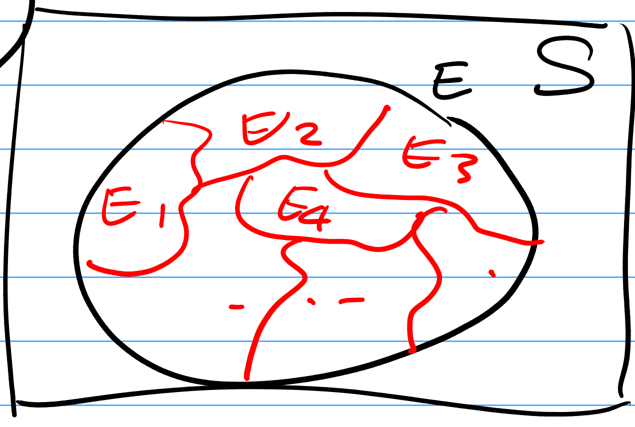
2. Unit-measure

$$P(S) = 1.$$

3. Countable additivity

If $(E_i)_{i=1}^{\infty}$ is a partition of E

$$\left(E_i E_j = \emptyset, \ \bigcup_{i=1}^{\infty} E_i = E\right)$$



then

$$P(E) = \sum_{i=1}^{\infty} P(E_i)$$

Comments

1) distributive Law

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

IF E_i disjoint.

2) This also holds for finite unions:

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i).$$

in particular: $A, B \subset S$ and $AB = \emptyset$ then

$$P(A \cup B) = P(A) + P(B).$$

Ex. Flip a coin:

$$S = \{H, T\}$$

what's a valid P on S?

$P(\{H\}) = \frac{1}{2}$ $\qquad$ $P(\overbrace{\{H, T\}}^{S}) = 1$

$P(\{T\}) = \frac{1}{2}$ $\qquad$ $P(\emptyset) = 0$

Does this sat. the K-axioms?

1. $P(E) \geq 0$? ✓
2. $P(S) = 1$? ✓
3. $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$ for disjoint E_i

One example:

$E = S$, $E_1 = \{H\}$, $E_2 = \{T\}$

E_1, E_2 partition E

Does additivity hold?

$$1 = P(S) = P(E) = P(E_1) + P(E_2)$$
$$= \frac{1}{2} + \frac{1}{2}$$
$$= 1 \quad \checkmark$$

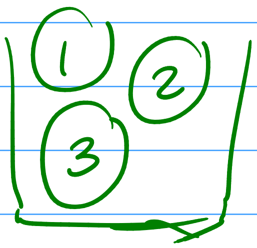
Ex. Could also have

$$P(S) = 1, \quad P(\emptyset) = 0$$

$$P(\{H\}) = \alpha \qquad P(\{T\}) = 1 - \alpha$$

for $0 \le \alpha \le 1$.

Ex.



$$S = \{1, 2, 3\}$$

$$P_1 = \frac{1}{4}, \quad P_2 = \frac{1}{4}, \quad P_3 = \frac{1}{2}$$

(non-neg and sum to 1)

$$P(\{1,2\}) = P_1 + P_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$P(\{1, 3\}) = p_1 + p_3 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

Theorem: Finite Sample Spaces

If $S = \{s_1, \dots, s_n\}$, $|S| = n < \infty$

and we choose some p_i, $i = 1, \dots, n$

so that

① $p_i \geq 0$

② $\sum_{i=1}^{n} p_i = 1$.

then a valid prob. function on S

is

$P(E) =$ sum up p_i corresp. to each $s_i \in E$

$$= \sum_{i: s_i \in E} p_i .$$

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pf.

① $P(E) \geq 0 \quad \forall E \subset S$

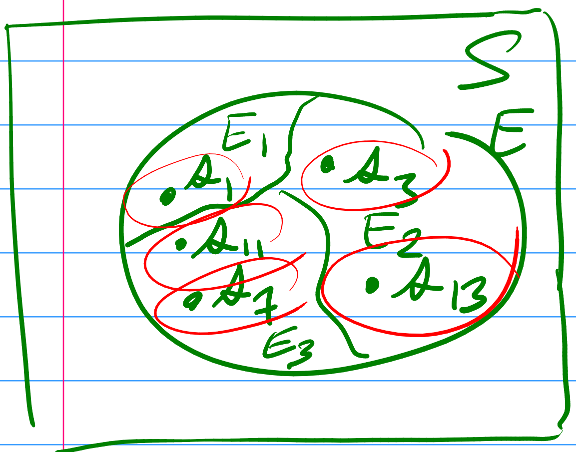
$$P(E) = \sum \overset{\geq 0}{p_i} \geq 0$$

② $P(S) = 1$

$$P(S) = \sum_{i:\, s_i \in S} p_i = \sum_{i=1}^{n} p_i = 1$$

③ If E_i partition E then

$$P(E) = \sum_{i=1}^{\infty} P(E_i).$$



Argue: $P(E) = P(E_1) + P(E_2) + P(E_3)$

p_1

$p_3 + p_{13}$

$p_7 + p_{11}$

$p_1 + p_3 + p_7 + p_{11} + p_{13}$

Theorem: $\boxed{P(\emptyset) = 0}$.

pf. $S = S \cup \emptyset \cup \emptyset \cup \emptyset \cup \emptyset \cup \cdots$

So by axiom 3

$$P(S) = P(S) + P(\emptyset) + P(\emptyset) + P(\emptyset) + \cdots$$

So

$$\sum_{i=1}^{\infty} P(\emptyset) = 0.$$

So $P(\emptyset) = 0.$