Lecture 2: Probability

Defn: Sample Space S

The set of possible outcomes.

Ex. Flip a coin.

$$S = \{H, T\}$$

Ex. Roll a six-sided die.

$$S = \{1, 2, 3, \ldots, 6\}$$

Ex. Roll two dice:

$$S = \{(1,1), (1,2), (2,1), \ldots, (6,6)\}$$

Ex. Waiting time for bus to arrive

$$S = [0, \infty)$$

Ex. Number of customer arriving

$$S = \mathbb{N}_0 = \{0, 1, 2, 3 \ldots \}$$

Types of sample spaces:

1. finite: $|S| < \infty$
2. infinite:
   1. countable
   2. uncountable

Defn: Outcome

We call an element $\underbrace{s}_{\text{outcome}} \in \underbrace{S}_{\text{sample space.}}$ an "outcome"

Defn: Event

An "event" E is a subset of S

$$\underbrace{E}_{\text{event}} \subset \underbrace{S}_{\text{sample space}}$$

equiv. $\subseteq$

Ex. $S = \{1, \ldots, 6\}$

then $E = \{1,2\} \subset S$

event that I roll a 1 or a 2.

Ex. $S = \{(i,j) : 1 \le i \le 6, 1 \le j \le 6\}$

$E = \{(1,2), (3,2)\} \subset S$

roll 1 then 2

OR

roll 3 then 2

$F = \{(1,2), (2,3)\} \subset S$

$E \neq F$.

We say an event "occurs" if the observed outcome of experiment is in E.

Ex. $S \subset S$

So S is an event.

event that something happens

Ex. $\emptyset \subset S$

So $\emptyset$ is an event.

event that nothing happens??

Axiomatic Probability

Given a sample space S

Want: For every event E, want to assign some measure of the prob. of E occurring

Mathematically:

For each $E \subset S$ we assign

$P(E)$

prob fn.

prob of E

What are rules for P?

(1) mathematically consistent

(2) encode some intuitions about prob.

Defn: Probability Function

Given a sample space S a prob. fn P is a function

$$P : 2^S \to \mathbb{R}$$

that satisfies the Kolmogorov Axioms

(1) non-negative:

$$P(E) \geq 0 \quad \forall E \subset S$$

(2) unit-measure

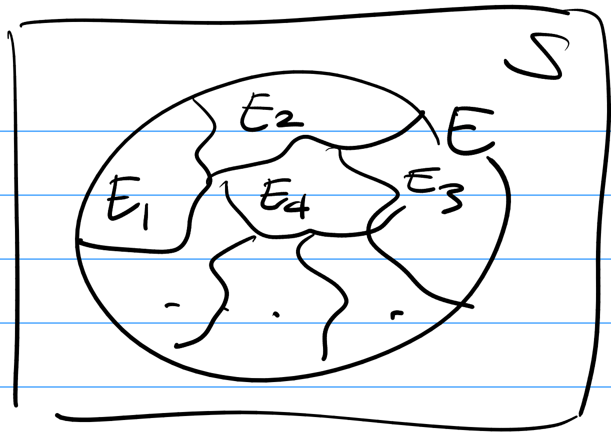
$$P(S) = 1.$$

(3) Countable additivity

If $(E_i)_{i=1}^{\infty}$ is a partition of E

$\left(E_i E_j = \emptyset \text{ for } i \neq j \right.$

$\left. \text{and } \bigcup_{i=1}^{\infty} E_i = E \right)$



$$P(E) = \sum_{i=1}^{\infty} P(E_i).$$

Comments on Axiom 3

① distributive law

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

If E_i are disjoint.

② This also holds for finite partitions

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

If E_i disjoint.

In particular: $A, B \subset S$ and $AB = \emptyset$
then $P(A \cup B) = P(A) + P(B)$.

Ex. Flip a coin: $S = \{H, T\}$

what's a valid P on S?

$P(\{H\}) = \frac{1}{2}$ $\quad P(\overbrace{\{H, T\}}^{S}) = 1$

$P(\{T\}) = \frac{1}{2}$ $\quad P(\emptyset) = 0$

$\{H\} \cup \{T\}$

Does this sat K-axioms?

(1) $P(E) \geq 0 \quad \forall E \subset S$ ✓

(2) $P(S) = 1$ ✓

(3) $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$ for disjoint E_i

One example of (3)

$E = S$, $E = E_1 \cup E_2$

$E_1 = \{H\}$, $E_2 = \{T\}$.

$$1 = P(S) = P(E) = P(E_1) + P(E_2)$$
$$= \frac{1}{2} + \frac{1}{2} = \underline{1}$$

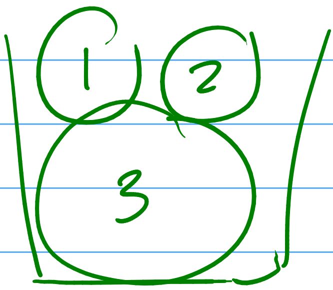
<u>Ex.</u> $S = \{H, T\}$

$P(S) = 1 \qquad P(\{H\}) = \alpha$

$P(\emptyset) = 0 \qquad P(\{T\}) = 1 - \alpha$

for $0 \le \alpha \le 1$.

<u>Ex.</u>



$S = \{1, 2, 3\}$

$p_1 = \frac{1}{4}, \; p_2 = \frac{1}{4}, \; p_3 = \frac{1}{2}$

(non-neg and sum to 1)

$P(\{1, 2\}) = p_1 + p_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$P(\{1, 3\}) = p_1 + p_3 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

Theorem: Finite Sample Spaces

Let $S = \{s_1, \ldots, s_n\}$, $|S| = n < \infty$.

Choose some p_i, $i = 1, \ldots, n$ so

that ① $p_i \geq 0$

② $\sum_{i=1}^{n} p_i = 1$

then a valid prob. function is

$$P(E) = \text{sum } p_i \text{ corresp. to } s_i \in E$$

$$= \sum_{i : s_i \in E} p_i$$

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pf. Show that this sats K-axioms:

① $P(E) \geq 0 \quad \forall E \subset S$

$$P(E) = \sum_i \overbrace{p_i}^{\geq 0} \geq 0$$

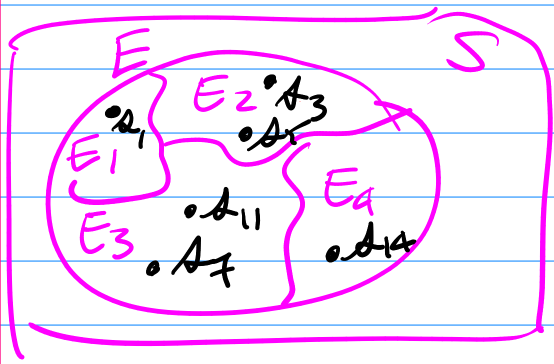
② $P(S) = 1.$

$$P(S) = \sum_{i:\, s_i \in S} p_i = \sum_{i=1}^{n} p_i = 1$$

③ $(E_i)_{i=1}^{\infty}$ partition E then

$$P(E) = \sum_{i=1}^{\infty} P(E_i).$$

Sketch:



$$P(E) = \overset{p_1}{P(E_1)} + \overset{p_3+p_5}{P(E_2)} + \overset{p_7+p_{11}}{P(E_3)} + \overset{p_{14}}{P(E_4)}$$

$= p_1 + p_3 + p_5 + p_7 + p_{11} + p_{14}$

Basic Theorems:

$$\boxed{\text{Theorem: } P(\emptyset) = 0}$$

pf. $S = S \cup \emptyset \cup \emptyset \cup \emptyset \cup \cdots$

$P(S) = P(S) + P(\emptyset) + P(\emptyset) + \cdots$

$\downarrow$

$$\sum_{i=1}^{\infty} P(\emptyset) = 0$$

must be that $P(\emptyset) = 0$