Lecture 3: Basic Theorems

Theorem: Finite Additivity

Axiom 3: $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$

if E_i disjoint

Finite Add: $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$

if E_i disjoint.

Pf. $n=2$ i.e.

$$E = A \cup B \quad \text{where } AB = \emptyset$$

$$= A \cup B \cup \emptyset \cup \emptyset \cup \emptyset \cup \cdots$$

$$P(E) = P(A \cup B \cup \emptyset \cup \emptyset \cdots)$$

$$= P(A) + P(B) + \underbrace{P(\emptyset) + P(\emptyset) + \cdots}_{0}$$

$$= P(A) + P(B)$$

For $n > 2$ use induction.

Ex. E = "its raining"

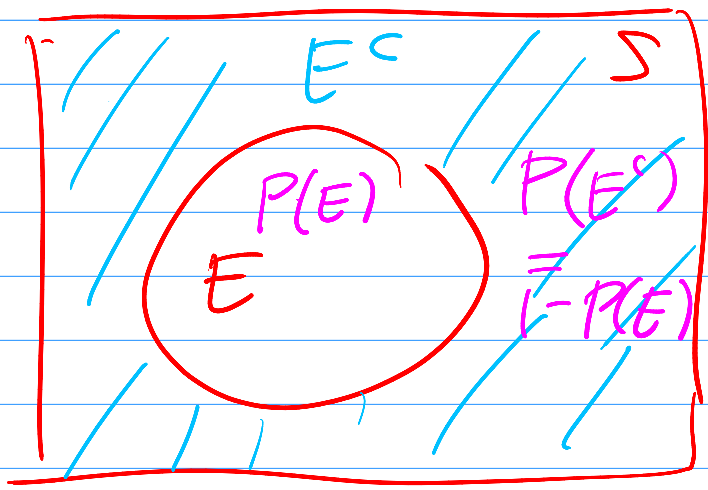
$P(E) = 1/3$

$P(\text{"not raining"}) = P(E^c) = 2/3$

$= 1 - P(E)$

Theorem: $P(E^c) = 1 - P(E)$

$P(S) = 1$



pf. $S = E \cup E^c$

← disjoint union

So by (Finite) additivity

$$1 = P(S) = P(E) + P(E^c)$$

So, re-arrange,

$$P(E^c) = 1 - P(E).$$

$$\boxed{\text{Theorem: } 0 \le P(E) \le 1}$$

pf. $P(E) \ge 0$ true by Axiom I

Also: $P(E^c) \ge 0$

so $1 - P(E) \ge 0$

so $P(E) \le 1$.

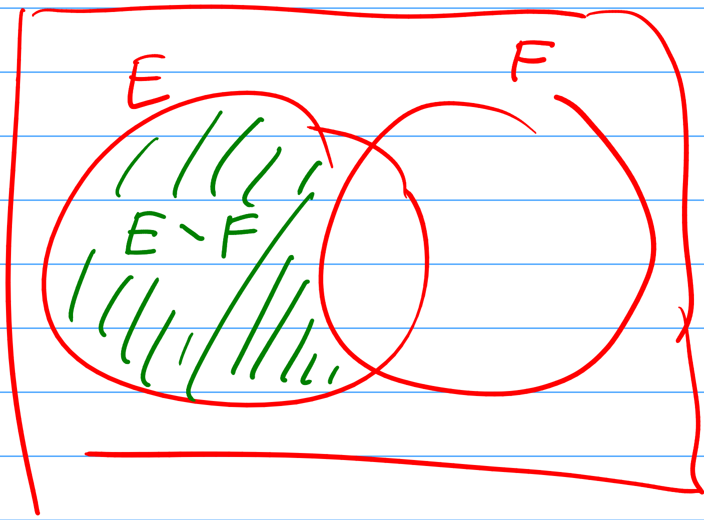
Theorem: If $E, F \subset S$ then

$$P(E \setminus F) = P(E) - P(EF)$$

pf.

$E \setminus F = EF^c$

$\underbrace{E = EF \cup EF^c}_{\text{partition}}$



So

$P(E) = P(EF) + P(EF^c)$ (additivity)

then re-arrange:

$$P(EF^c) = P(E) - P(EF)$$

$$\parallel$$

$$P(E \setminus F).$$

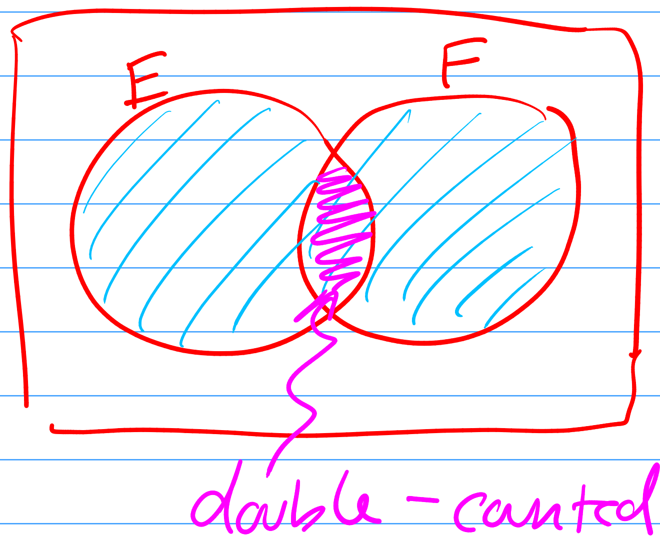
<u>Theorem:</u> $E, F \subset S$

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

Pf. $E \cup F$

$= E \cup FE^c$

(partition)



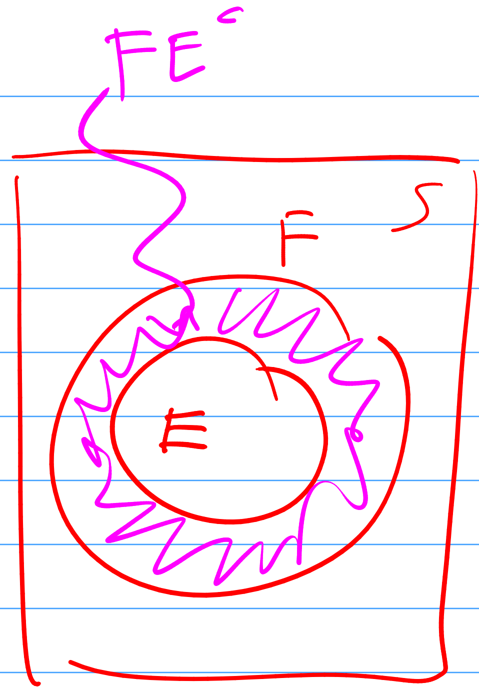
so

$$P(E \cup F) = P(E) + P(FE^c)$$

$$= P(E) + \underbrace{P(F) - P(EF)}_{\text{prev. thrm}}.$$

Theorem: $E \subset F \subset S$

then $P(E) \le P(F)$.



pf.

Axiom 1: $P(FE^c) \ge 0$

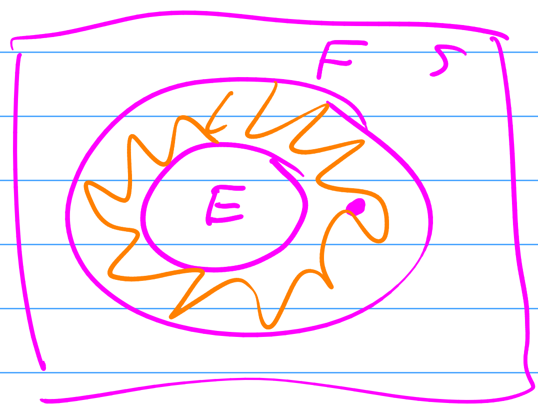
$P(F) - P(FE) \ge 0$

So, $P(F) \ge P(\underbrace{FE}_{E})$ since $E \subset F$

So, $P(F) \ge P(E)$.

If $E \subset F$ but $E \ne F$ (proper subset)

$\cancel{P(E) < P(F)}$?



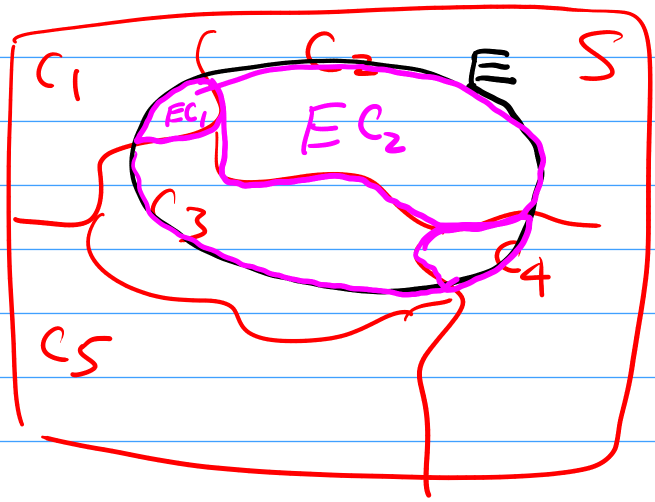
Could be that

$P(F \setminus E) = 0$

even if $F \setminus E \ne \emptyset$

Theorem: let (C_i) are a partition of S and let $E \subset S$

$$P(E) = \sum_i P(EC_i).$$



pf:

① (EC_i) partition E

(i) $EC_i \cap EC_j = \emptyset \quad \forall i \neq j$

(ii) $E = \bigcup_i EC_i$

② $P(E) = P(\bigcup_i EC_i)$

$$= \sum_i P(EC_i).$$

Equally likely Outcomes

I have a sample space S

$$S = \{s_1, s_2, \ldots, s_n\}$$

finite.

Assume that each outcome is equally likely:

$$P(\{s_i\}) = P(\{s_j\}) \quad \forall i, j.$$

then

$$P(\{s_i\}) = 1/n \quad \text{where } n = |S|$$

Reason:

disjoint union

$$1 = P(S) = P\left(\bigcup_{i=1}^{n} \{s_i\}\right)$$

$$= \sum_{i=1}^{n} P(\{s_i\})$$

$$= n P(\{s_1\})$$

So $P(\{s_1\}) = 1/n$.

More generally:

$$P(E) = \frac{\#\text{ outcomes in } E}{\#\text{ outcomes in } S} = \frac{|E|}{|S|}.$$

Ex. Roll a six-sided die

$$S = \{1, 2, \ldots, 6\}$$

If all rolls are equally likely

let $E = \{1, 2\}$

then $P(E) = \frac{|E|}{|S|} = \frac{2}{6} = 1/3.$

Counting

Ex. An experiment has 3 factors:

1. 2 temp settings
2. 2 pressure settings
3. 4 humidity settings

Q: How many configs can I run the experiment?

A: $16 = 2 \cdot 2 \cdot 4$

Fundamental Theorem of Counting

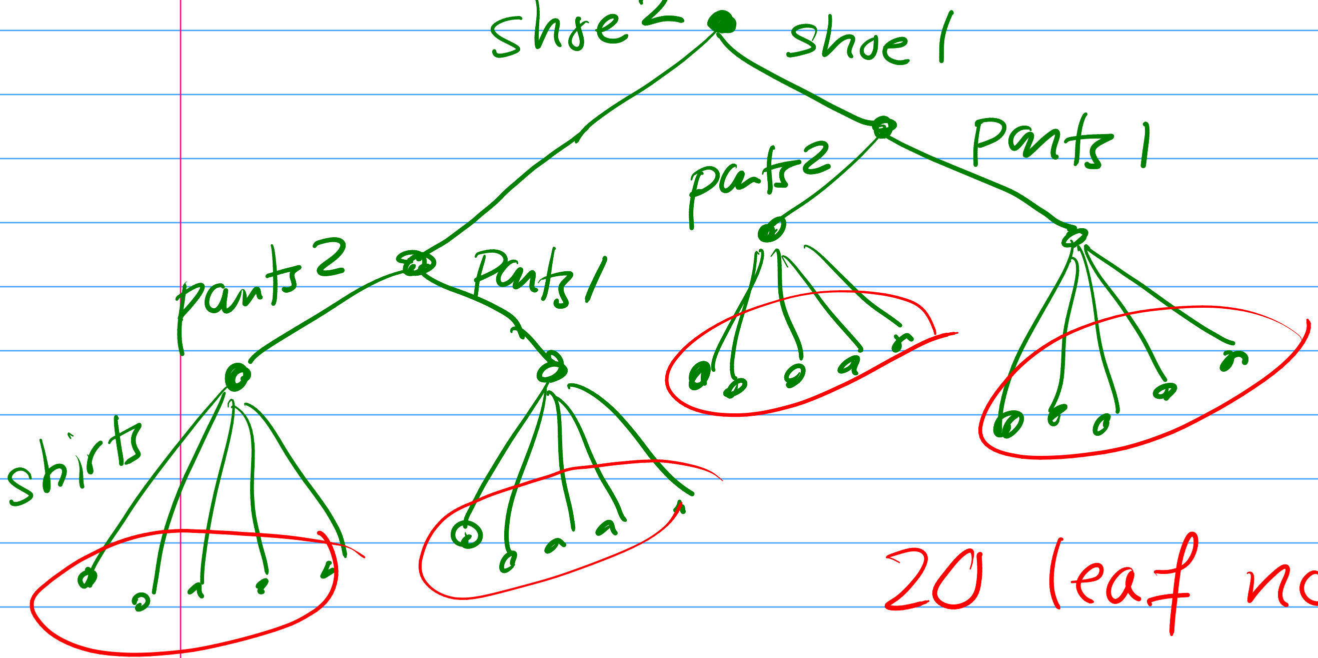
- Task consists of k sub-tasks
- Subtask i has n_i options
- I can choose options for each task independently of the others

The number of ways to complete the overall task is

$$N = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$
$$= \prod_{i=1}^{k} n_i$$

Ex. Man has 5 shirts, 2 pair pants, 2 pair shoes.
How many outfits does he have?

By FTC : $20 = 5 \cdot 2 \cdot 2$



20 leaf nodes

<u>Ex</u>. I have a deck of 52 cards

I shuffle them so each ordering is equally likely.

<u>Q</u>: What's the prob that the cards are in <u>order</u>?

$\longrightarrow$ A - K, C, D, H, S

E = cards are in order

S = all possible orderings

then $P(E) = \frac{|E|}{|S|}$.

So $|E| = 1$

To count $|S|$, use FTC

task#	task	# ways
1	Choose card 1	52
2	" 2	51
3	" 3	50
⋮	⋮	⋮
52	" 52	1

So FTC says:

$$|S| = 52 \cdot 51 \cdot 50 \cdot 49 \cdots 3 \cdot 2 \cdot 1$$

So

$$P(E) = 1 / 52 \cdot 51 \cdot 50 \cdots 3 \cdot 2 \cdot 1$$

Defn: Factorial

For any non-neg. integer n we define "n factorial" as

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$
$$= \prod_{i=1}^{n} i$$

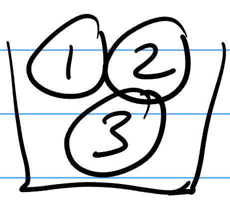
Defn: $0! = 1$

Ex. In prev. example,

$$P(E) = 1/52!$$

Sampling w/ and w/o Replacement/Ordering

Order:



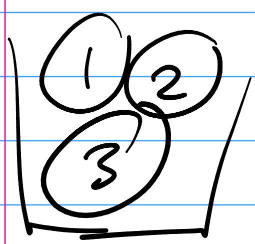
draw 1: 1 3 2

draw 2: 2 1 3

Q: Are these different?

w/ ordering: Yes
w/o ordering: No

Replacement:



Can I draw an item twice?

Ex. Can I draw 1 1 2 ?

w/ replacemnt: Yes
w/o replacemnt: No

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4 Options:

	w/o repl.	w/ repl.
ordered	1	2
unordered	4	3