Lecture 3: Basic Theorems

Theorem: Finite Additivity

Third Axiom: $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$

for E_i disjoint

Finite Additivity:

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

for E_i disjoint.

$\underline{n=2}$

Pf. $E = A \cup B$ where $AB = \emptyset$

$E = A \cup B \cup \emptyset \cup \emptyset \cup \emptyset \cup \cdots$

countable partition of E

Axiom 3:

$$P(E) = \underbrace{P(A) + P(B)} + \underbrace{P(\emptyset) + P(\emptyset) + \cdots}_{P(\emptyset) = 0}$$

For $n > 2$: use induction.

Ex. E = "its raining"

$$P(E) = 1/3$$

$$P(\underbrace{\text{"not raining"}}_{E^c}) = 2/3$$

$$= 1 - 1/3$$

$$= 1 - P(E)$$

$$\boxed{\textbf{Theorem: } P(E^c) = 1 - P(E)}$$

Pf. $S = \underbrace{E \cup E^c}_{\text{partition}}$

By additivity:

$$\underset{\uparrow \text{ Axiom 2}}{1} = P(S) = P(E) + P(E^c)$$

by rearranging: $P(E^c) = 1 - P(E)$.

$\boxed{\text{Theorem: } 0 \le P(E) \le 1}$

pf. $P(E) \geq 0$ by Axiom 1

$P(E^c) \geq 0$ //

so $1 - P(E) \geq 0$

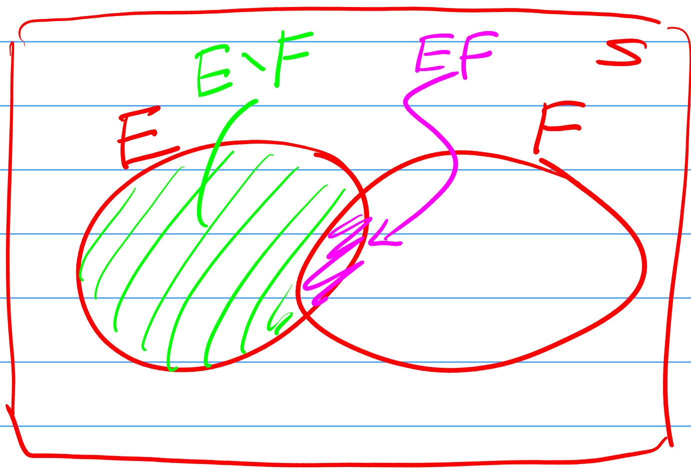
rearrange to get $P(E) \leq 1$.

Theorem: $E, F \subset S$ then

$$P(E \setminus F) = P(EF^c)$$
$$= P(E) - P(EF)$$

pf.

$$E = \underbrace{EF \cup EF^c}_{\text{partition of } E}$$



so

$$P(E) = P(EF) + P(EF^c)$$

So $$P(EF^c) = P(E) - P(EF)$$

$$\|$$

$$P(E \setminus F)$$

Theorem: If $E, F \subset S$ then

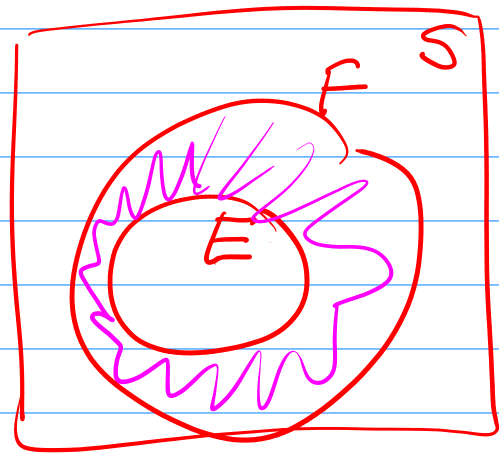
$$P(E \cup F) = P(E) + P(F) - P(EF)$$

Pf. $$E \cup F = \underbrace{E \cup FE^c}_{\text{partition}}$$

$$\begin{aligned} P(E \cup F) &= P(E \cup FE^c) \\ &= P(E) + P(FE^c) \\ &= P(E) + P(F) - P(FE) \end{aligned}$$

<u>Theorem:</u> $E \subset F \subset S$

then $P(E) \leq P(F)$.



pf. By Axiom,

$$P(FE^c) \geq 0$$

and

$$P(F) - P(FE) \geq 0$$

So

$$P(F) \geq P(FE) = P(E).$$

What if $E \subset F$ but $E \neq F$ (proper subset)

then

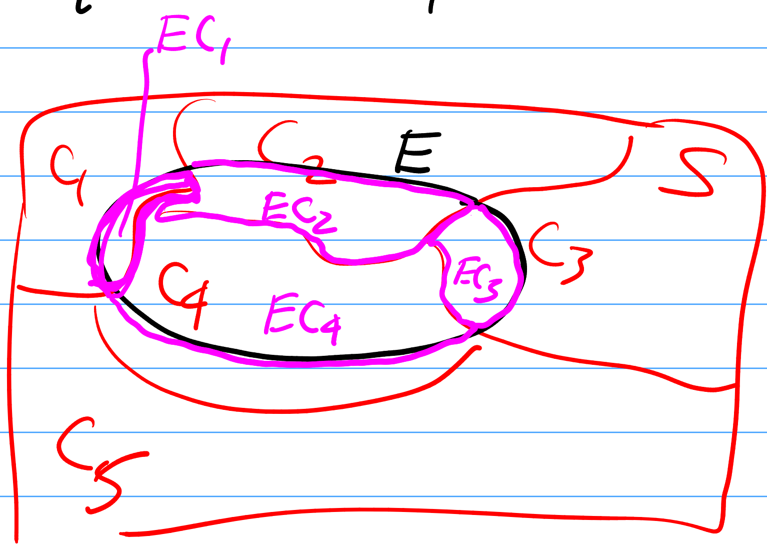
$$\cancel{P(E) < P(F)} \;?$$

Could be that $P(FE^c) = 0$

$$(P(\{a\}) = 0)$$

Theorem: If (C_i) are a partition of S and let $E \subset S$ then

$$P(E) = \sum_i P(EC_i)$$



pf.

(1) Show (EC_i) partition E

(i) $EC_i \cap EC_j = \emptyset \quad \forall i \neq j$

(ii) $E = \bigcup_i EC_i$.

(2) So by Additivity:

$$P(E) = P\left(\bigcup_i EC_i\right)$$

$$= \sum_i P(EC_i).$$

Equally Likely Outcomes

Consider a sample space S

$$S = \{s_1, s_1, \ldots, s_n\}$$

where $n = |S| < \infty$.

assume that

$$P(\{s_i\}) = P(\{s_j\}) \quad \forall i, j$$

then

$$P(\{s_i\}) = 1/n .$$

Reason: $S = \bigcup_{i=1}^{n} \{s_i\}$

So

$$1 = P(S) = \sum_{i=1}^{n} P(\{s_i\})$$

$$= n P(\{s_1\})$$

So $P(\{a_i\}) = 1/n$.

More generally:

$$P(E) = \frac{\#\text{ outcomes in } E}{\#\text{ outcomes in } S} = \frac{|E|}{|S|}.$$

Ex. Roll a six-sided die

$$S = \{1, 2, \ldots, 6\}$$

and all rolls equally likely then

if $E = \{1, 2\}$

then $P(E) = \dfrac{|E|}{|S|} = \dfrac{2}{6} = 1/3.$

Ex. An experiment has 3 factors:

(1) 2 temp settings
(2) 2 pressure settings
(3) 4 humidity settings

Q: How many experiments are possible?

[Can set any of settings independently]

A: $2 \cdot 2 \cdot 4 = 16$

Fundamental Theorem of Counting (FTC)

I have a task w/

- k subtasks
- n_i options for i^{th} task
- we can do tasks independently

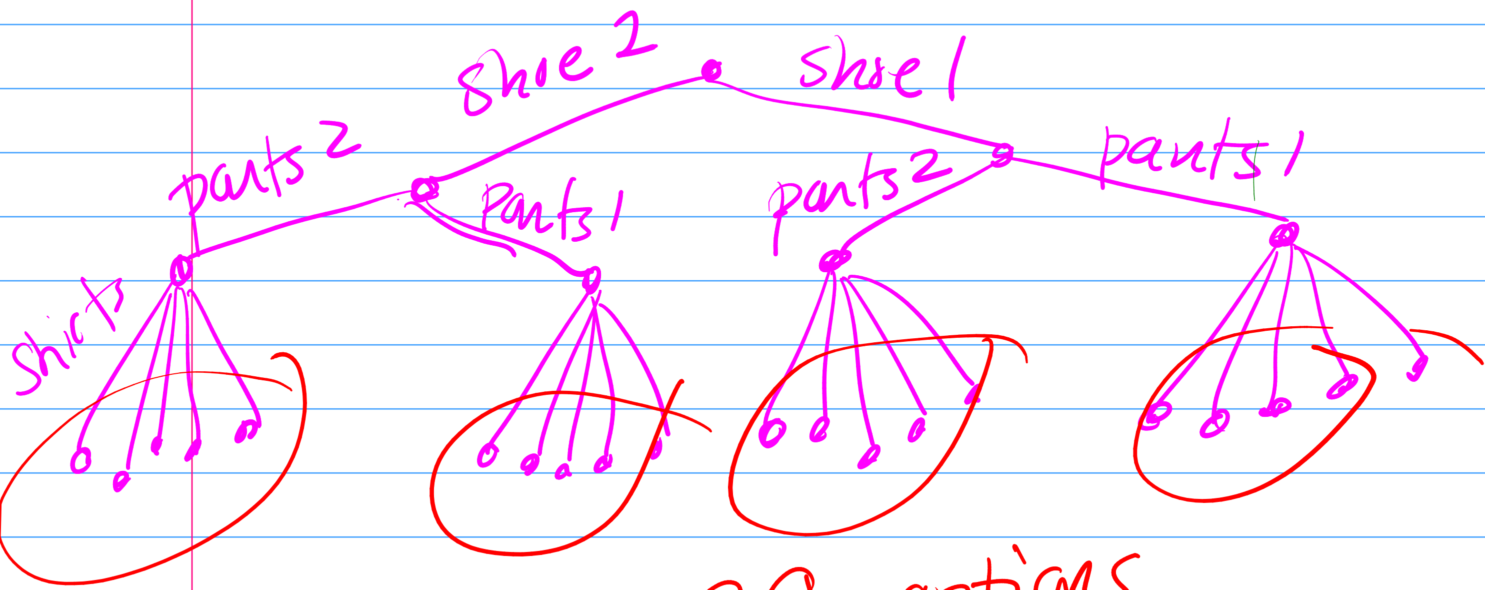
Then the number of ways to accomplish the overall task is

$$N = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$
$$= \prod_{i=1}^{k} n_i$$

ex. Man has 5 shirts, 2 pair pants, 2 pair shoes.

How many outfits does he have?

By FTC $5 \cdot 2 \cdot 2 = 20$.



20 options

ex. 52 card deck.

Shuffle so every ordering is equally likely.

Q: what's the prob the cards are in order? → A-K, C, D, H, S

E = cards in order

S = all possible shuffles.

So $$P(E) = \frac{|E|}{|S|}$$

$|E| = 1$

Count $|S|$, use FTC:

task #	# ways
1	52
2	51
3	50
$\vdots$	$\vdots$
52	1

by FTC: $|S| = 52 \cdot 51 \cdot 50 \cdot \ldots \cdot 3 \cdot 2 \cdot 1$

So $P(E) = \frac{1}{52 \cdot 51 \cdots 3 \cdot 2 \cdot 1}$

Defn: Factorial

For any non-neg. integer n, we define "n factorial" as

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = \prod_{i=1}^{n} i$$

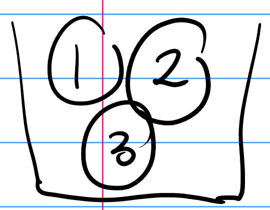
$0! = 1$

---

ex: (prev.) then $P(E) = 1/52!$

Sampling w/ and w/o Replacement/Order

Order



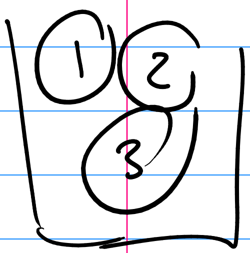
draw 1:	draw 2:
1 3 2	3 2 1

Q: Are these considered different samples?

A: Yes ⇝ sampling w/ order mattering

No ⇝ sampling w/o order

Replacement



Q: Can I sample an item more than once?

e.g. get a sample 1 1 2

A: Yes ⇝ sampling w/ replacement

No ⇝ sampling w/o replacement

	w/o repl.	w/ repl.
ordered	1	2
unordered	4	3