Lecture 4: Counting

Defn: Permutation

A permutation is an ordering of objects.

Ex. 1 2 3

permutations

$$\left.\begin{matrix} 1,2,3 & 1,3,2 \\ 2,1,3 & 2,3,1 \\ 3,1,2 & 3,2,1 \end{matrix}\right\} 6 \text{ perms} = 3!$$

Theorem: The number of ways to permute r items is $r!$

pf. Use FTC w/ r tasks

task	# ways
Choose 1^{st}	r
Choose 2^{nd}	$r-1$
" 3^{rd}	$r-2$
⋮	⋮
" r^{th}	$\underline{1}$

$$r(r-1)\cdots 3\cdot 2\cdot 1 = r!$$

Theorem: Sampling w/o replacement w/ ordering

If I have n items and I sample r,

1. w/o repl
2. w/ ordering

The number of ways to do this is

$$\frac{n!}{(n-r)!}.$$

pf. Use FTC

task	# ways
Sample 1st	n
" 2nd	$n-1$
" 3rd	$n-2$
⋮	⋮
" rth	$n-r+1$

$$n(n-1)(n-2)\cdots(n-r+1)$$

$$= \frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)\cancel{(n-r)\cdots 3\cdot 2\cdot 1}}{\cancel{(n-r)\cdots 3\cdot 2\cdot 1}}$$

Ex. I form a committee from $10^{=n}$ students of size $3.^{=r}$

The committee has:

(1) president, (2) VP, (3) treasurer

How many ways can I form this?

w/ order: $1^{st} \to$ pres
$2^{nd} \to$ VP
$3^{rd} \to$ treasurer

w/ repl: same person can't two roles

Use our rule: $\frac{n!}{(n-r)!}$

$$\frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}} = 10 \cdot 9 \cdot 8$$
$$= 720$$

Ex. Lotto.

Basket w/ 25 numbered balls

draw 4 of them,

(assume all such draws equally likely)

Guess: 1 3 22 7

Prob I win? $E =$ "I win"

$$P(E) = \frac{|E|}{|S|} = \frac{1}{|S|}$$

$$|S| = \frac{25!}{(25-4)!} = \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot \cancel{21!}}{\cancel{21!}}$$

$$= 25 \cdot 24 \cdot 23 \cdot 22$$

thus

$$P(E) = \frac{1}{|S|} = \frac{1}{25 \cdot 24 \cdot 23 \cdot 22}.$$

Theorem: Sampling w/ repl and w/ Ordering

The num of ways to sample r from n

- w/ repl
- w/ ordering

is n^r.

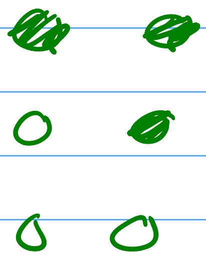
Pf. Use FTC

task	# ways
choose 1st	n
⋮	n
⋮	⋮
" r^{th}	n

$$\underbrace{n \cdots n}_{} = n^r$$

Ex. Braille

Six spots – each raised or not.



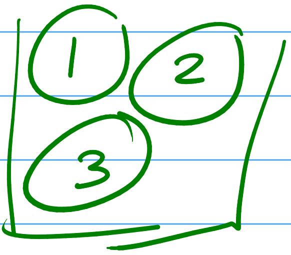
Q: How many braille "letters" are possible?

Sampling $r = 6$ spots from $n = 2$ (raised or not) options.

By theorem: $2^6 = 64$

Consider sampling w/o replacement w/o order

Ex.



draw $r = 2$ from $n = 3$

If I did care about order:

(1, 2)	(1, 3)	(2, 3)
(2, 1)	(3, 1)	(3, 2)

$\frac{3!}{(3-2)!} = 6$

w/o order

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$ 3 ways

General Fact: Sampling w/o repl.

Each unordered sample of size r gives rise to $r!$ ordered samples.

Ex. $\{1,2,3\} \to (1,2,3), (3,2,1), (2,1,3), \ldots$ $\Big\}\ 6 = 3!$

Fact: [w/o replacement]

$$\underbrace{(\#\text{ ordered})}_{\frac{n!}{(n-r)!}} = r!\ \underbrace{(\#\text{ unordered})}_{???}$$

So $\#\text{ unordered} = \dfrac{n!}{r!(n-r)!}$

Theorem: Sampling w/o repl., w/o ordering

I can sample r from n

- w/o order
- w/o repl.

in $\binom{n}{r} = \dfrac{n!}{(n-r)!r!}$ ways

Binomial coefficient

read: "n choose r"

Ex. I have 10 $= n$ professors, how many ~~co-equal~~ unordered committees of size 4 $= r$ can I assemble?

w/o replacement.

$$\binom{n}{r} = \binom{10}{4} = \boxed{\frac{10!}{4!(10-4)!}}$$

$$= \frac{10!}{4!\,6!} = \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7 \cdot \cancel{6!}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{6!}}$$

$$= 10 \cdot 3 \cdot 7 = 210$$

Ex. I deal a 5-card poker hand. What's the prob that my hand is 4 aces and the 2 of clubs?

$E = \{4 \text{ aces} + 2 \text{ clubs}\}$

$|E| = 1$

Unordered: Card hands don't have order

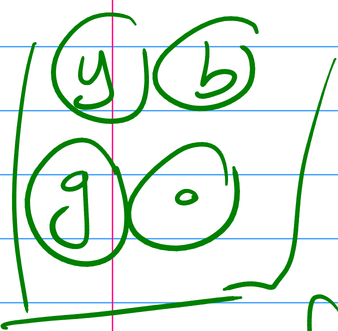
No replacemt.

$n = 52$, $r = 5$

$$|S| = \binom{52}{5} \approx 2.5 \text{ mil}$$

$$P(E) = \frac{1}{|S|} \approx \frac{1}{2.5\text{mil}}.$$

Ex. Jar w/ 4 marbles of colors yellow, blue, orange, green.



I choose 3 from jar w/o replacement.

[all such samples equally likely]

Q: what's the prob of choosing a (y) and (b) among ?

$$E = \{\{y,b,o\},\ \{y,b,g\}\}$$

$$\boxed{|E| = 2}$$

S = all such samples

$$|S| = \binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!} = \boxed{4}$$

So $P(E) = \frac{2}{4} = 1/2$.

Sampling w/ replacement, w/o ordering

Consider $n = 3$ and $r = 2$

Ordered: $n^r = 9$

$(1,2)$ $(2,1)$ | $(1,3)$ $(3,1)$ | $(2,3)$ $(3,2)$ | $(1,1)$ | $(2,2)$ | $(3,3)$

Unordered:

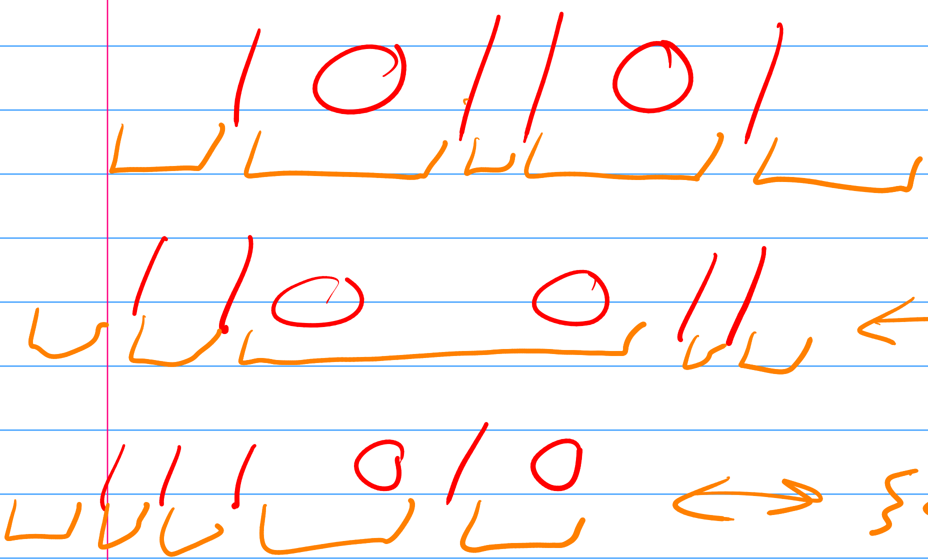
$\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $\{1,1\}$ | $\{2,2\}$ | $\{3,3\}$

6 ways

No $r!$ corresp. btwn
Ordered and unorderd
— when sampling w/ repl.

Game of Partitioning

How many ways can I partition $r = 2$ objects using $n - 1 = 4$ walls



$\leftrightarrow \{2, 4\}$

$\leftrightarrow \{3, 3\}$

$\leftrightarrow \{4, 5\}$

1-1 corresp btwn drawing and # ways to sample r from n w/ repl, w/o order.

I have $n-1+r$ symbols
so I can arrange in $(n-1+r)!$ ways.

Have to divid out non-unique arrangements!

- can switch any walls : $(n-1)!$
- " object : $r!$

unique games/drawings :

$$\frac{(n-1+r)!}{(n-1)!\,r!}$$

<u>Theoram:</u> Sampling w/o order, w/ repl.

The number of ways to sample

- w/o order
- w/ repl.

is

$$\frac{(n+r-1)!}{(n-1)!\,r!} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}.$$