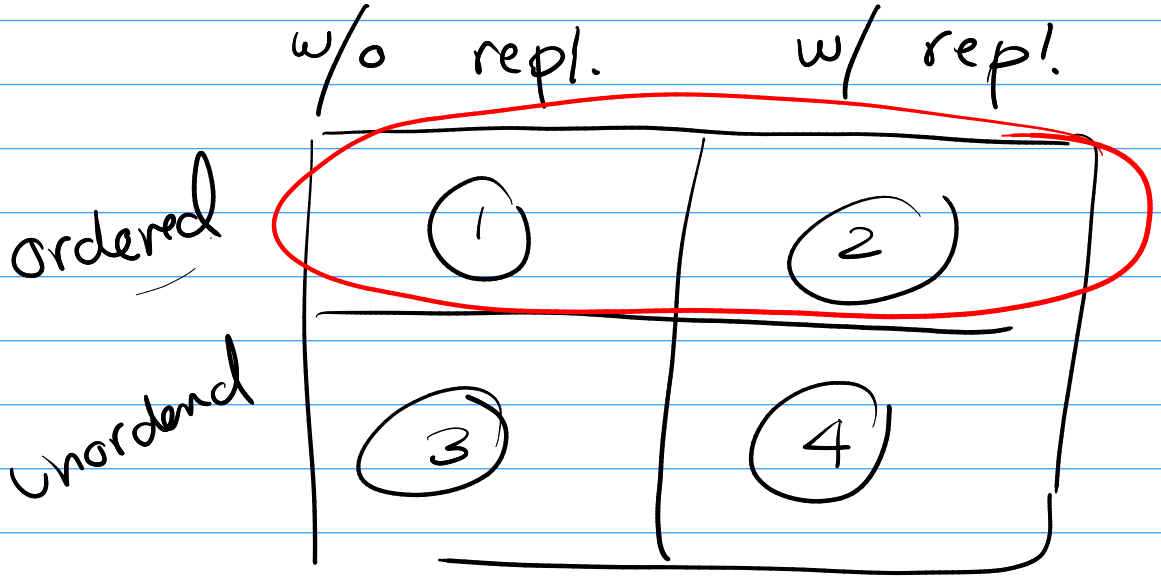
Lecture 4: Counting

4 options!



Defn: Permutation

A permutation is an ordering of objects.

Ex. (1) (2) (3)

perms:

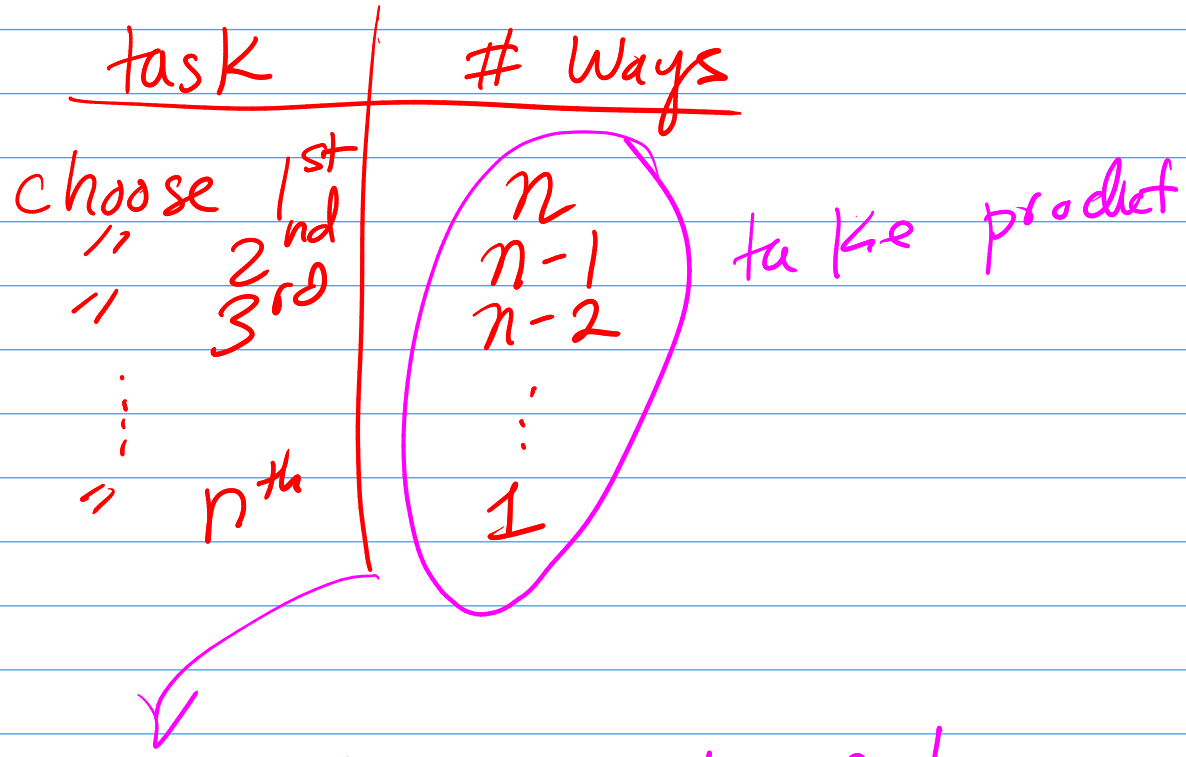
1, 2, 3 2, 1, 3 3, 1, 2

1, 3, 2 2, 3, 1 3, 2, 1

} 6 perms = 3!

Theorem: The number of ways to permute n items is $n!$

pf. Use FTC



Theorem: Sampling w/o repl, w/ ordering

The number of ways to sample r items from n

- w/o repl
- w/ ordering

is $n!/(n-r)!$.

Pf. Use FTC

task	# ways
Choose 1st	n
" 2nd	$n-1$
⋮	⋮
" r^{th}	$n-r+1$

product: $\boxed{n(n-1)(n-2)\cdots(n-r+1)}$

$$\frac{n!}{(n-r)!} = \frac{n(n-1)\cdots(n-r+1)\cancel{(n-r)\cdots 3\cdot 2\cdot 1}}{\cancel{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}}$$

Ex. I form a committee from $10 = n$ students of size $3 = r$ where the committee has

1. Pres
2. VP
3. Treasurer

How many committees can I form?

w/o repl. : one person can't hold multiple roles

w/ ordering: order = which role

$$\#\text{ ways: } \frac{n!}{(n-r)!} = \frac{10!}{(10-3)!}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}} = 10 \cdot 9 \cdot 8 = 720$$

Ex Lotto.

Basket w/ 25 numbered balls

draw 4 of them.

[all such draws equally likely]

Draw out 1 at a time, care about order.

Guess: (1)(3)(22)(7)

What's the prob I win?

E

Then $P(E) = \frac{|E|}{|S|}$

$|E| = 1$

$|S| = \frac{25!}{(25-4)!} = \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21!}{21!}$

So $P(E) = \frac{1}{25 \cdot 24 \cdot 23 \cdot 22}$.

Theorem: Sampling w/ repl, w/ ordering

The number of ways to sample r from n
- w/ repl
- w/ ordering

is n^r.

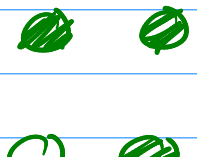
Pf:

task	# ways
choose 1st	n
" 2nd	n
⋮	⋮
" r^{th}	n

$\Big\}\ n^r$

Ex. Braille. $r=6$

Six spots: each raised or not. $n=2$

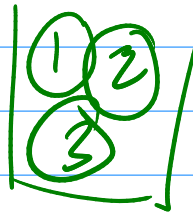


Q: How many braille characters are possible?

Then there are $2^6 = 64$ possibilities.

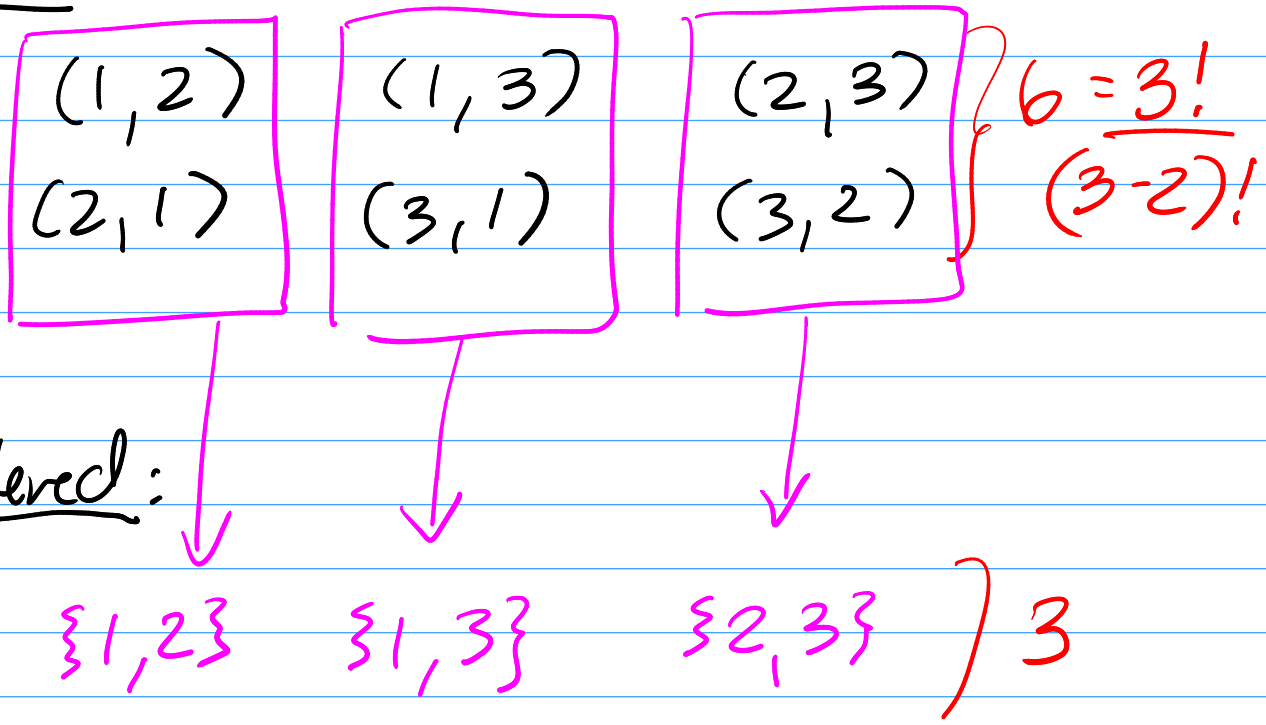
Sampling w/o replacement.

Ex.



Sample: $r=2$ from $n=3$

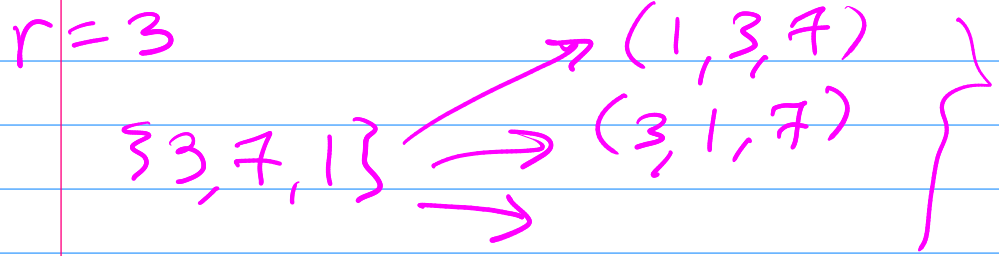
Order:



Unordered:

General Fact:

Each unordered sample of size r can be permuted in r! ways to make ordered samples.



Fact! (w/o replacement)

$$(\#\text{ ordered}) = r!\,(\#\text{ unordered})$$

$(\#\text{ ordered}) = n!/(n-r)!$

Theorem: Sampling w/o repl, w/o ordering

The number of ways to sample r from n

- w/o repl
- w/o ordering

is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

↑ Binomial coefficient
read: "n choose r"

Ex, I have $10 = n$ profs, how many co-equal committees of size $4 = r$ can I form?

unordered — w/o repl: can't have prof repeated

$$\boxed{\binom{n}{r} = \frac{10!}{4!(10-4)!}} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6!}}{4!\cancel{6!}} = \frac{10 \cdot \overset{3}{\cancel{9}} \cdot \cancel{8} \cdot 7}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2}}$$

$= 10,3 \cdot 7 = 210$

Ex. I deal a 5-card hand.
What's the prob that my hand is

$\underbrace{A, A, A, A, 2.}_{E}$

$$P(E) = \frac{|E|}{|S|}$$

$|E| = 4$

$|S| = \binom{52}{5} \approx 2.5 \text{ mil.}$

so $P(E) = \frac{4}{\binom{52}{5}} \approx \frac{4}{2.5 \text{ mil.}}$

Ex. Jar w/ 9 marbles of
Colors: yellow, blue, orange, green.
Choose 3 from Jar w/o repl.
[all such choices equally likely]

Q: what's the prob that (y) and (b) are among 3 choose.

$E = \{(y) \text{ and } (b)\}$

$E = \{\{y, b, o\}, \{y, b, g\}\}$

$\Rightarrow |E| = 2$

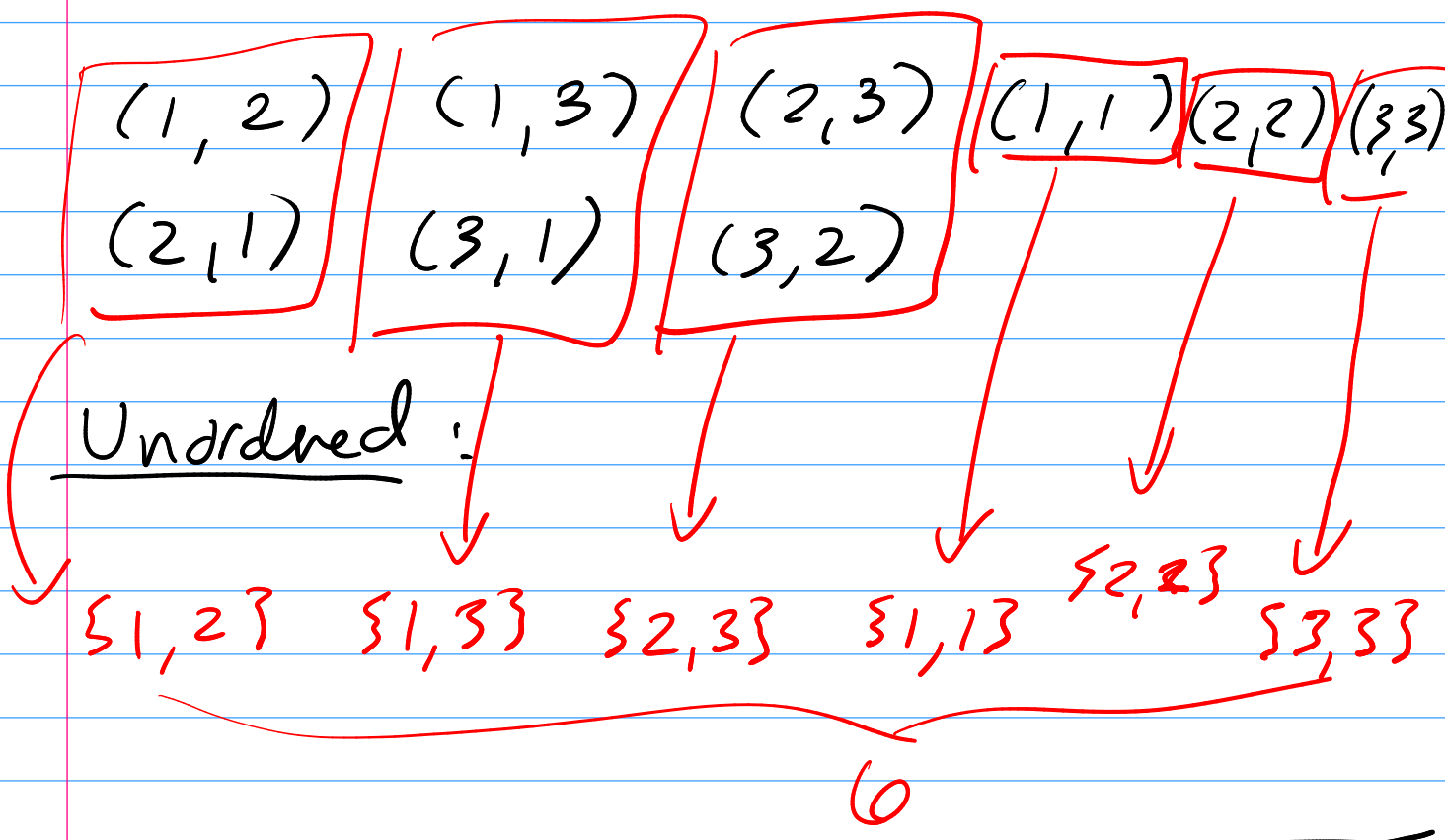
$$|S| = \binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4 \cdot 3 \cdot 2}{3 \cdot 2} = 4$$

So $P(E) = 2/4 = 1/2$.

Sampling w/ Replacement

$n=3,\ r=2$

Ordered: $n^r = 9$



Unordered:

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$ $\{1,1\}$ $\{2,2\}$ $\{3,3\}$

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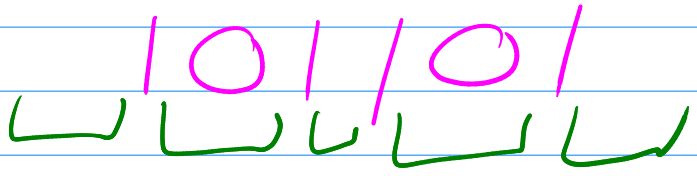
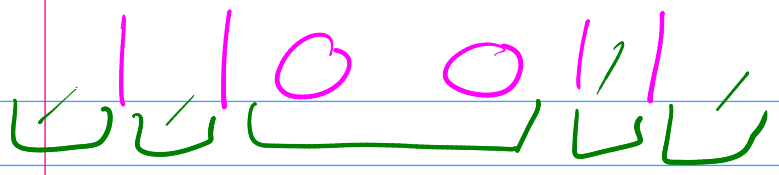
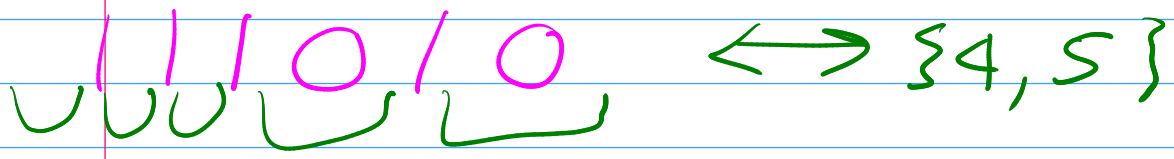
Game of Partitioning

How many ways can I partition $r=2$ objects using $n-1=4$ walls.



$\longleftrightarrow \{2,4\}$





1-1 corresp btwn symbol arrangements and samples.

How many unique arrangements can I make?

Overall: $r+n-1$ symbols

Can permute these in $(n+r-1)!$ ways

Need to divide out dups:

- permute walls around in $(n-1)!$
- " objs " $r!$

Number of distinct arrangements is

$$\frac{(n+r-1)!}{(n-1)!\,r!}$$

Theorem: Sampling w/ repl, w/o ordering

The number of ways to sample r from n

- w/ repl.
- w/o ordering

is

$$\frac{(n+r-1)!}{r!\,(n-1)!} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}.$$