Lecture 5: Conditional Prob

Ex, 10 passengers on a bus route w/ 5 stops.

Driver record # get off at each stop.

Q: How many possible records?

Stop	#
1	0
2	3
3	1
4	2
5	4

$\longleftrightarrow \{2,2,2,3,4,4,5,5,5,5\}$

$$\# \text{ ways} = \binom{n+r-1}{r} = \binom{5+10-1}{10}$$

$$= \binom{14}{10} = 1001$$

Ex, Jar w/ 4 marbles: y, b, o, g

Draw $r=3$ from $n=4$

(w/ repl, w/o order)

Q: what is the prob I get a y and b among 3 draw?

E

$$P(E) = \frac{|E|}{|S|}$$

$$E = \{\{y,b,o\},\{y,b,g\},\{y,b,y\},\{y,b,b\}\}$$

$$|E| = 4$$

$$|S| = \binom{n+r-1}{r} = \binom{4+3-1}{3} = \binom{6}{3} = 20$$

$$P(E) = 4/20 = 1/5.$$

	w/o repl	w/ repl.
ordered	$\frac{n!}{(n-r)!}$	n^r
Unordered	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$	$\binom{n+r-1}{r}$

Ex. Flip a coin twice.

What is the prob. of getting a H and T.

Option 1: S unordered

$$S = \{HH, TT, HT\}$$

$E = \{HT\}$

So $P(E) = |E|/|S| = 1/3$.

Option 2: Ordered Way

$$S = \{HH, TT, HT, TH\}$$

$E = \{HT, TH\}$

$$P(E) = 2/4 = 1/2.$$

Point of counting:

$$P(E) = \frac{|E|}{|S|} \leftarrow$$ outcomes in S equally likely

General Rule:

If I build S through a seq of independent actions then typically an ordered S makes sense.

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when sampling w/ repl. this matters.

when sampling w/o repl. often doesn't

$$P(E) = \frac{|E|\cancel{r!}}{|S|\cancel{r!}}$$

Conditional Prob.

ex. Survey W&M student,
ask about political afil and sex.

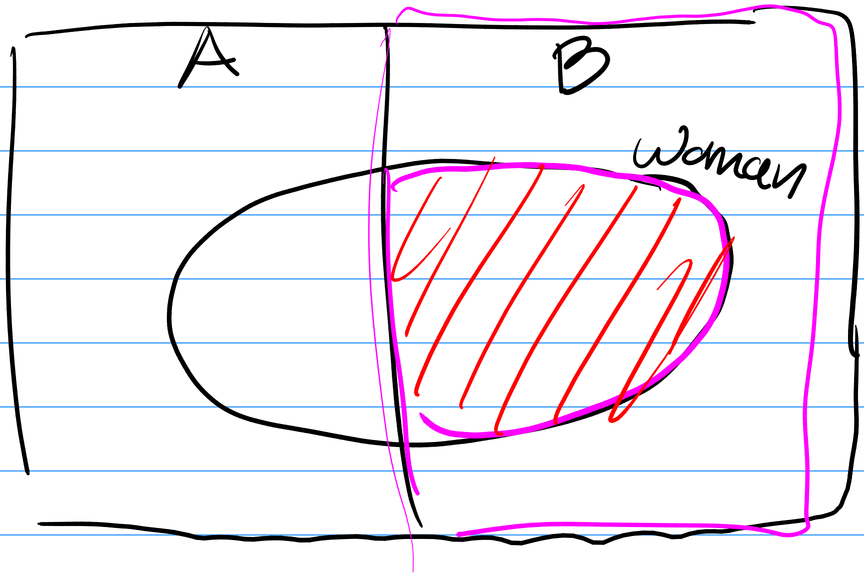
		A	B	
sex	men	501	238	739
	women	782	123	905
			361	1644

Q1: If randomly sample student, what's the prob. they are a woman?

$$P(\text{Woman}) = 905/1644$$

Q2: Given student is in B what's prob they are a woman?

$$P(\text{Woman } \underline{\text{given}} \text{ B}) = 123/361$$



Q1: $P(\text{woman}) = \dfrac{\bigcirc}{\square}$

Q2: $P(\text{woman given } B) = \dfrac{D}{\square}$

$$= \frac{\bigcirc \cap \boxed{B}}{\boxed{B}}$$

Defn: Conditional Prob

If $A, B \subset S$ and $P(B) > 0$ then the conditional prob of A given B is

read: "given"

$$P(A \mid B) = P(AB) / P(B).$$

Fact: (1) $P(B|B) = 1$

Pf: $P(B|B) = \dfrac{P(BB)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$.

(2) If $AB = \emptyset$ then $P(A|B) = 0$.

pf: $P(A|B) = \dfrac{P(AB)}{P(B)} = \dfrac{P(\emptyset)}{P(B)} = \dfrac{0}{P(B)} = 0$.

Ex. Roll two dice.

Q: What's prob. $\underbrace{\text{the first is } 2}_{A}$ given $\underbrace{\text{the sum is} \leq 5}_{B}$.

$$P(A|B) = P(AB)/P(B)$$

$$= \frac{|AB|/|S|}{|B|/|S|}$$

$= |AB| / |B|$

first

second	1	2	3	4	5	6
1	O	O X	O	O		
2	O	O X	O			
3	O	O X				
4	O	X				
5		X				
6		X				

"O" = B

"X" = A

$3/10$.

Theorem: Compound Prob.

If $P(A) > 0$ and $P(B) > 0$

then $P(AB) = P(A|B)P(B)$

$= P(B|A)P(A)$.

Pf $P(A|B) = \dfrac{P(AB)}{P(B)}$

rearrange: $P(AB) = P(A|B)P(B)$.

Recall: partitioning theorem

If (A_i) partition S then

$$P(B) = \sum_i P(BA_i).$$

Theorem: Law of Total Prob.

If (A_i) partition S and $P(A_i) > 0$ then $B \subset S$ we have

$$P(B) = \sum_i P(B|A_i)P(A_i)$$

pf.

$$P(B) = \sum_i P(BA_i) \quad \text{[partition thrm]}$$

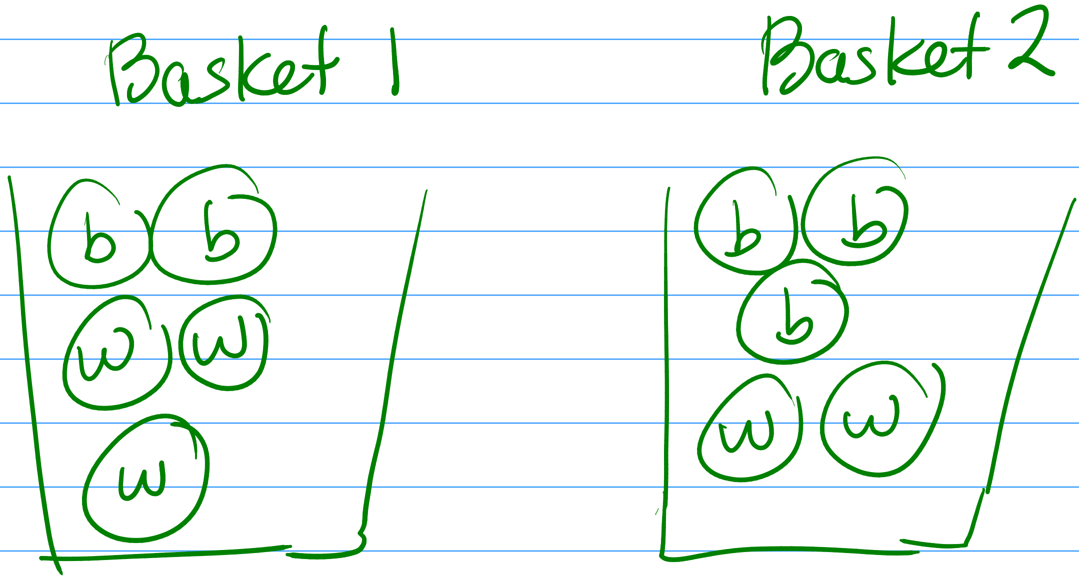
$$= \sum_i P(B|A_i)P(A_i) \quad \text{[compound prob.]}$$

Special Case: A and A^c partition S

Law of total prob:

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c).$$

Ex.



Game: (1) randomly select ball from basket 1 and place in basket 2

(2) randomly select ball from basket 2

Q: What's prob I select black ball on step 2?

W = choose (w) on step 1

W^c = " (b) " "

B = " (b) on step 2

B^c = " (w) on step 2

Want: $P(B)$

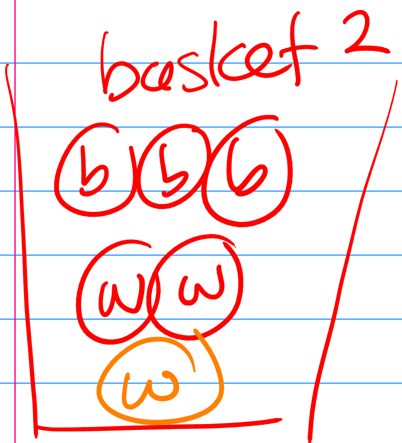
Solve: Condition on W

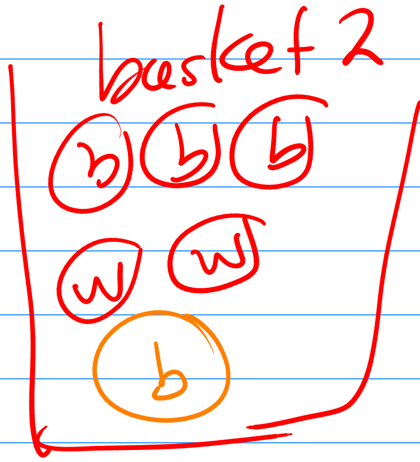
Law of total prob:

$$P(B) = P(B|W)P(W) + P(B|W^c)P(W^c)$$

$$P(W) = 3/5, \quad P(W^c) = 2/5$$

$$P(B|W) = 1/2 \qquad P(B|W^c) = 2/3$$





$$P(B|W)P(W) + P(B|W^c)P(W^c)$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{3}{5}\right) + \left(\tfrac{2}{3}\right)\left(\tfrac{2}{5}\right)$$

$$= 17/30$$

Theorem: Bayes' Theorem

How to get $P(A|B)$ from $P(B|A)$?

If $A, B \subset S$, $P(A), P(B) > 0$ then

$$P(A|B) = P(B|A)\frac{P(A)}{P(B)}.$$

Pf:

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}.$$

Ex. Cont. prev.

Given I choose a black ball on second step, what's the prob I chose white on first?

By Bayes' we have that

$$P(W|B) = \frac{P(B|W)P(W)}{P(B)}$$

$$= \frac{\left(\frac{1}{2}\right)\left(3/5\right)}{\left(17/30\right)}.$$

Theorem: Law of Tot. Prob + Bayes'

If (A_i) partition S, $P(A_i) > 0$, $P(B) > 0$

then

$$P(A_i \mid B) = \frac{P(B \mid A_i)\,P(A_i)}{\sum_j P(B \mid A_j)\,P(A_j)}$$

pf:

$$P(A_i \mid B) = \frac{P(B \mid A_i)\,P(A_i)}{P(B)} \quad \text{[Bayes']}$$

$$= \frac{P(B \mid A_i)\,P(A_i)}{\sum_j P(B \mid A_j)\,P(A_j)} \quad \text{[total prob]}$$

Special Case: A and A^c partition S

$$P(A \mid B) = \frac{P(B \mid A)\,P(A)}{P(B \mid A)\,P(A) + P(B \mid A^c)\,P(A^c)}.$$