Lecture 5: Conditional Prob.

Ex. 10 $=r$ passengers on bus route w/ $n=5$ stops.

Driver record # people that exit at each stop.

Q: How many possible records are there?

Stop	#
1	0
2	3
3	1
4	2
5	4

$\longleftrightarrow \{2,2,2,3,4,4,5,5,5,5\}$

$$\binom{n+r-1}{r} = \binom{5+10-1}{10} = \binom{14}{10} = 1001$$

Ex. Jar w/ 4 marbles: y, b, o, g

Draw $r=3$ from $n=4$.

(w/ repl., w/o ordering)

Q: Prob $\underbrace{\text{I get a y and b}}_{E}$?

$$P(E) = \frac{|E|}{|S|}$$

$$E = \{\{y,b,o\}, \{y,b,g\}, \{y,b,b\}, \{y,b,y\}\}$$

$$|E| = 4$$

S = all such samples,

$$|S| = \binom{n+r-1}{r} = \binom{4+3-1}{3} = \binom{6}{3} = 20$$

So $P(E) = 4/20 = 1/5$.

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	w/o repl.	w/ repl.
ordered	$n!/(n-r)!$	n^r
unordered	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$	$\binom{n+r-1}{r}$

Ex. Flip coin twice.

what's prob of getting H and T?

Option 1: Unordered

$$S = \{HH, TT, HT\}$$

$E = \{HT\}$

So $P(E) = \dfrac{|E|}{|S|} = 1/3$

Option 2: ordered

$$S = \{HH, TT, TH, HT\}$$

$E = \{HT, TH\}$

$P(E) = 2/4 = 1/2$.

Point of counting:

$P(E) = \dfrac{|E|}{|S|}$ ← equally likely

General rule:

If I build S through a seq. of independent actions then typically an ordered S makes sense.

when sampling w/ repl. this matters. Often doesn't matter when sampling w/o repl.

$$P(E) = \frac{|E|\cancel{r!}}{|S|\cancel{r!}}$$

Ex. Survey W&M students

- favorite dessert: Icecream
  Cake
- living: on campus / not

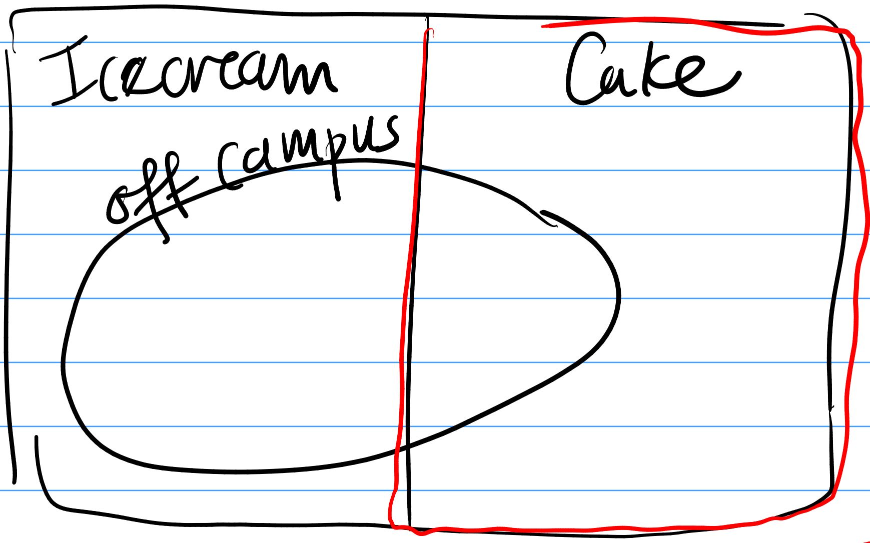
	Icecream	Cake	
On Camp	501	238	739
off Campus	782	123	905
		361	1644

Q1: Select student, what's prob. they live off campus?

$$P(\text{off campus}) = 905/1644.$$

Q2: Given that student likes Cake, what's prob they live off campus?

$$P(\text{off given Cake}) = 123/361.$$



Q1: $P(\text{off campus}) = \dfrac{\bigcirc}{\square}$

Q2: $P(\text{off given Cake}) = \dfrac{D}{\square}$

$= \dfrac{\bigcirc \cap \square}{\square}$

Defn: Conditional Prob

If $A, B \subset S$ and $P(B) > 0$ then the conditional prob of A given B

read: given

$$P(A \mid B) = \frac{P(AB)}{P(B)}.$$

Fact: (1) $P(B \mid B) = \underline{1}$

Pf. $P(B|B) = \frac{P(BB)}{P(B)} = \frac{P(B)}{P(B)} = 1.$

(2) If $AB = \emptyset$ then $P(A|B) = 0$.

Pf $P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(\emptyset)}{P(B)} = \frac{0}{P(B)} = 0.$

Ex, Roll two dice.

Q: What's the prob. the $\boxed{\text{first is } 2}^{A}$ given $\boxed{\text{sum is} \leq 5.}_{B}$

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$= \frac{|AB| / \cancel{|S|}}{|B| / \cancel{|S|}}$$

$$= \frac{|AB|}{|B|}$$

second \ first	1	2	3	4	5	6
1	O	O X	O	O		
2	O	O X	O			
3	O	O X				
4	O	X				
5		X				
6		X				

O = B

X = A

$= 3/10$

Theorem: Compound Prob.

$$P(AB) = P(A|B)P(B)$$
$$= P(B|A)P(A)$$

pf. $P(A|B) = P(AB)/P(B)$

rearrange: $P(A|B)P(B) = P(AB)$

Recall partitioning theorem:

If (A_i) partition S then

$$P(B) = \sum_i P(BA_i).$$

Theorem: Law of Total Prob.

If (A_i) partition S then $B \subset S$,

$$P(B) = \sum_i P(B|A_i)\,P(A_i).$$

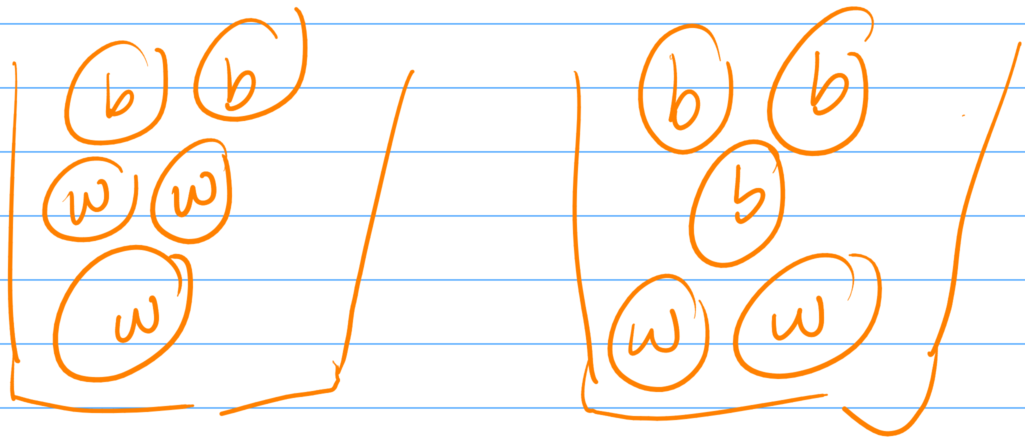
Pf. $P(B) = \sum_i P(BA_i)$ [partitioning]

$= \sum_i P(B|A_i)P(A_i)$ [compound prob.]

Special Case: A and A^c partition S

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$$

ex, Basket 1 Basket 2



Step 1: select ball from basket 1, place in basket 2

Step 2: randomly select ball from basket 2

Q: what's prob of getting b on step 2?

B_1 = choose b step 1

B_1^c = " w "

B_2 = choose b step 2

B_2^c = " w "

Want: $P(B_2)$.

Soln: : Condition on B_1.

Law of total prob:

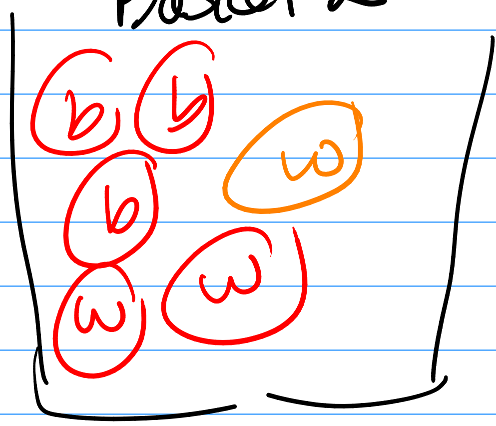
$$P(B_2) = P(B_2 | B_1)P(B_1) + P(B_2 | B_1^c)P(B_1^c)$$

$$P(B_1) = 2/5 \quad , \quad P(B_1^c) = \frac{3}{5}$$

$$P(B_2 | B_1) = 4/6 = \frac{2}{3} \qquad P(B_2 | B_1^c) = \frac{1}{2}$$

Basket 2

Basket 2



$$P(B_2) = P(B_2 | B_1)P(B_1) + P(B_2 | B_1^c)P(B_1^c)$$

$$= \left(\frac{2}{3}\right)\left(\frac{2}{5}\right) + \left(\frac{1}{2}\right)\left(3/5\right) = 17/30$$

Theorem: Bayes' Theorem

Go from $P(A|B)$ to $P(B|A)$.

$$P(A|B) = P(B|A)\frac{P(A)}{P(B)}.$$

Pf. $$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}.$$

Ex. Cont. prev.

B_2

Given choose b on step 2,

what's prob I chose w step 1?

B_1^c

Use Bayes'

$$P(B_1^c|B_2) = \frac{P(B_2|B_1^c)P(B_1^c)}{P(B_2)}$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{3}{5}\right)\Big/\left(\tfrac{17}{30}\right)$$

Theorem: Law total prob + Bayes

(A_i) partition S then

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_j P(B|A_j)P(A_j)}$$

Pf. $$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)} \quad \text{[Bayes']}$$

$$= \frac{P(B|A_i)P(A_i)}{\sum_j P(B|A_j)P(A_j)} \quad \text{[total prob]}.$$

Special Case: A and A^c partition S

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$