Lecture 7: Random Variables

We'd like to say:

$$P(X=1) \leftarrow \text{formally: abuse of notation}$$

X = # heads among 3 flips of coin,

$$P(X=1) = P(\{HTT, THT, TTH\})$$

"$X=1$" short-hand for $\underbrace{\{s \in S : X(s)=1\}}_{\subseteq S}$ $= X^{-1}(\{1\})$

inverse image of 1 under X.

More generally: $A \subset \mathbb{R}$ then

$$P(X \in A) \overset{\text{def}}{=} P(X^{-1}(A))$$

$$= P(\{s \in S : X(s) \in A\})$$

Ex. $P(X = 1 \text{ or } 2)$

$$= P(X \in \{1,2\})$$
$$= P(X^{-1}(\{1,2\}))$$
$$= P(\{HHT, THH, TTH, THT, HTT\})$$
$$= 6/8.$$

Defn: Support of a RV (for now)

The support of a RV is all the values it can take on.

(range of X).

Ex. $\text{Support}(X) = \{0,1,2,3\}$.

notice: $P(X=5) = 0$

generally: $A \subset \mathbb{R}$, $A \cap \text{Support}(X) = \emptyset$

then $P(X \in A) = 0.$

Defn: Discrete and Continuous RVs.
(for now)

1. discrete RVs : support is finite/countable

   Ex. X = sum of two dice

   Ex. X = number of customers arriving

2. continuous RVs: support is not countable

   Ex. time/space

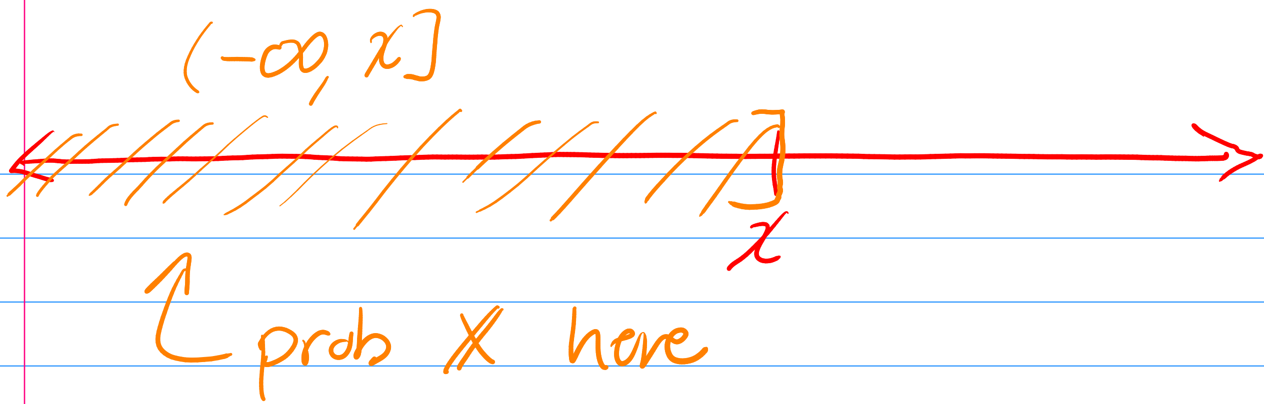
Defn: Cumulative Distribution Function (CDF)

If X is a RV then its CDF is a function

$$F : \mathbb{R} \to \mathbb{R}$$

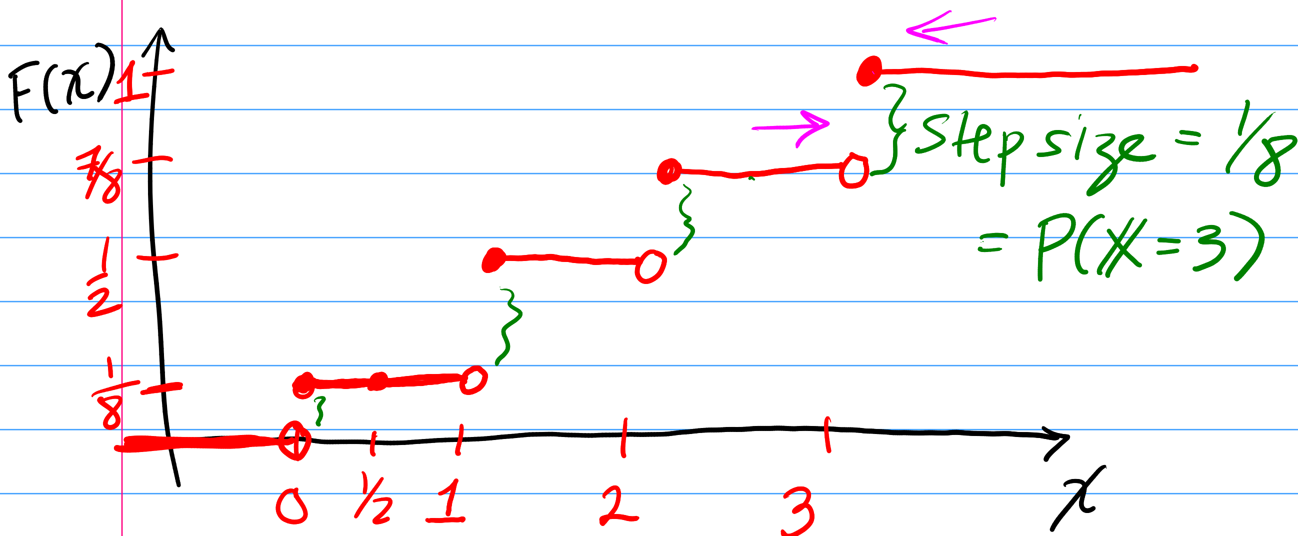
defined for $x \in \mathbb{R}$ as

$$F(x) = P(X \le x)$$



Formally: $F(x) = P(X \in (-\infty, x])$.

Ex. Toss a coin 3 times,
X = # heads



$$F(0) = P(X \le 0) = P(X=0) = 1/8$$

$$F(\tfrac{1}{2}) = P(X \le \tfrac{1}{2}) = P(X=0) = 1/8$$

$$F(1) = P(X \le 1) = P(X=0) + P(X=1) = \tfrac{1}{2}$$

$$F(1.5) = P(X \le 1.5) = P(X \le 1) = 1/2$$

$$F(2) = P(X \le 2) = 7/8$$

$F(3) = P(X \le 3) = 1$

$F(4) = P(X \le 4) = 1$

$F(-1) = P(X \le -1) = 0$

Note: (1) Jumps in CDF at values in support

(2) Step function.

Facts:

(1) $\boxed{0 \le F(x) \le 1}$

Pf. $F(x) = P(\sim\sim) \in [0,1]$

(2) $\boxed{\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1.}$

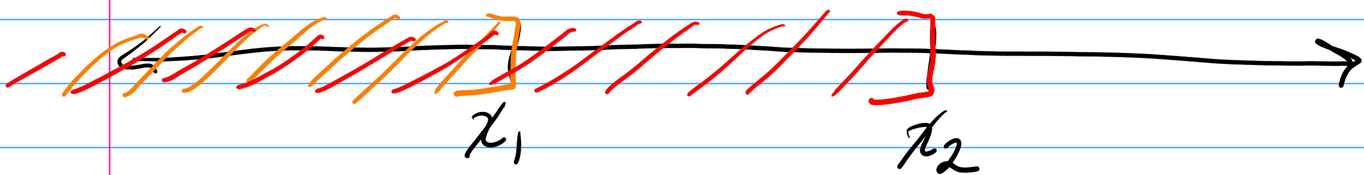
(3) $\boxed{F \text{ is non-decreasing}}$

If $x_1 < x_2$ then $F(x_1) \le F(x_2)$.

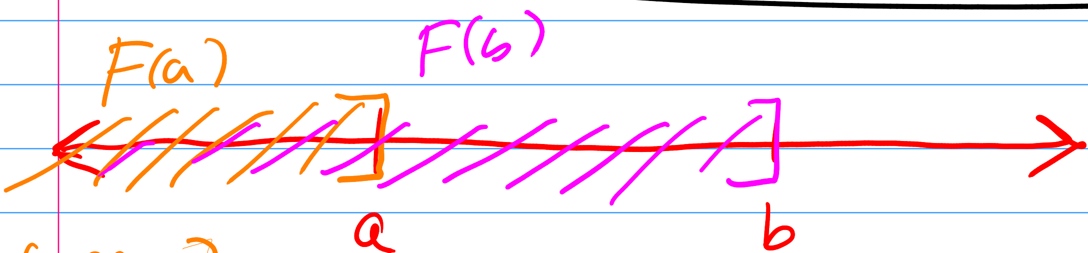
$$F(x_1) = P(X \le x_1)$$
$$= P(X \in \underbrace{(-\infty, x_1]})$$
$$= P(X^{-1}((-\infty, x_1]))$$

$\le$

$$F(x_2) = \cdots = P(X^{-1}(\underbrace{(-\infty, x_2]}))$$



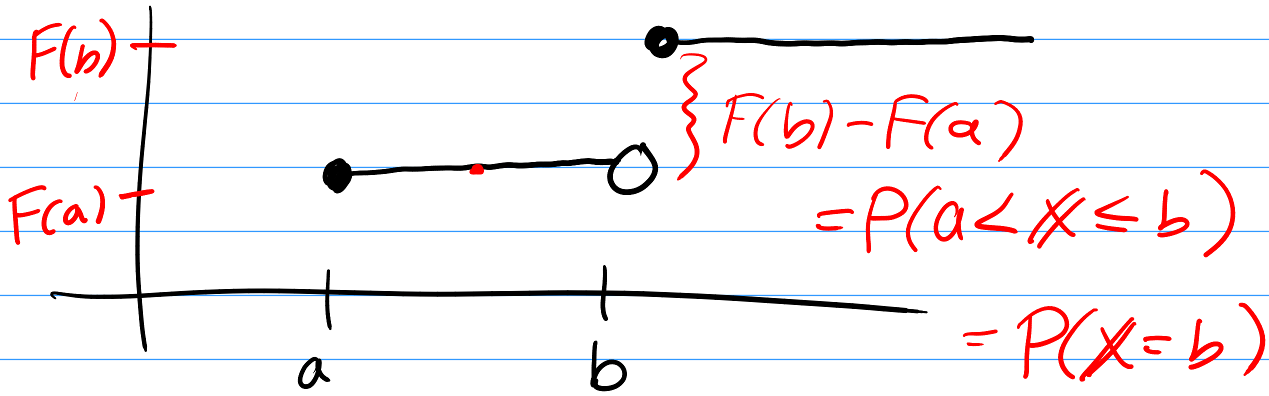
$$\boxed{④ \quad P(a < X \le b) = F(b) - F(a). \qquad a \le b}$$



$$\underbrace{(-\infty, a]}_{(-\infty, b]}$$

$$(-\infty, b] \setminus (-\infty, a] = (a, b].$$

practical:



(5) $\boxed{F \text{ is right-continuous}}$

Recall: cts function: $\lim_{x \to a} F(x) = F(a)$.

Right-Continuous:

$$\lim_{x \to a^+} F(x) = F(a).$$

Note: cts fn is right cts.

Theorem:

F is the CDF of some RV

iff

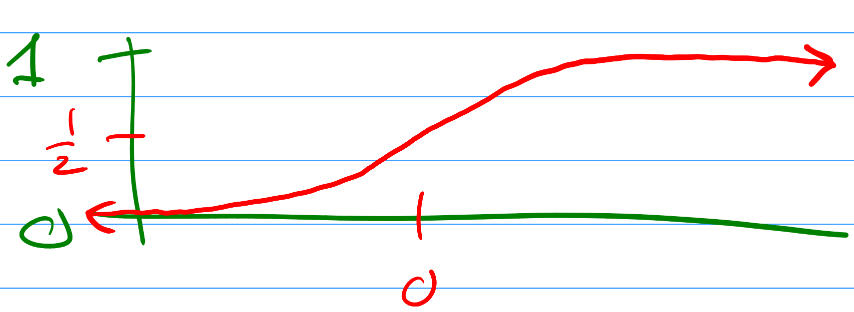
(1) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$

(2) F is non-decreasing

(3) F is right cts.

Ex, let

$$F(x) = \frac{1}{1+e^{-x}} \quad \text{for } x \in \mathbb{R}.$$



Q: Is this a valid CDF?

Check 3 conditions:

(1) $\lim_{x \to -\infty} F(x) = \frac{1}{1+e^{-(-\infty)}} = 0$

$\lim_{x \to \infty} F(x) = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$

(2) <u>non-decreasing?</u>

$\frac{dF}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} > 0$

(3) <u>Right-continuous?</u>

Diff'able, thus cts, thus right-cts.

<u>Defn:</u> <u>Identical Distribution</u>

We say two RVs X and Y are equal in dist if

$\forall A \subset \mathbb{R}$

$$P(X \in A) = P(Y \in A).$$

We write: $X \stackrel{d}{=} Y$.

This doesn't mean $X = Y$ (as fns).

Ex. Flip 3 coins:

$$X = \#\text{ heads}$$

$$Y = \#\text{ tails}$$

these are different RVs:

$$X(HHT) = 2$$

$$Y(HHT) = 1$$

However, $X \stackrel{d}{=} Y$

$$P(X=0) = \frac{1}{8} = P(Y=0)$$

$$P(X=1) = 3/8 = P(Y=1)$$

$$\vdots$$

Theorem: $X \stackrel{d}{=} Y$ iff $F_X = F_Y$.

CDF of X

Ex. Toss a coin (indeply) until a H appears.

$$S = \{H, TH, TTH, TTTH, \dots\}$$

Let p be the prob. I get a H on any flip.

X = # flips until I get a H

$s \in S$	$X(s)$
H	1
TH	2
TTH	3
TTTH	4
⋮	⋮

$F(x) = P(X \le x)$

Q: what's the CDF?

Look at $P(X = x)$

Let H_i = i^{th} toss is a H

$T_i = H_i^c$

$P(X=i) = P(T_1 T_2 T_3 \cdots T_{i-1} H_i)$

$= P(T_1)P(T_2)\cdots P(T_{i-1})P(H_i)$

$= (1-p)(1-p)\cdots(1-p)p$

$= (1-p)^{i-1}p$