Lecture 7: Random Variables

We'd like to say

$$P(X=1)$$ slightly abusive

what we really mean is

$$P(X=1) = P(\{HTT, THT, TTH\}) = 3/8$$

"$X=1$" short-hand for:

$$\{s \in S : X(s) = 1\} \subset S$$

inverse image of $\{1\}$ under X

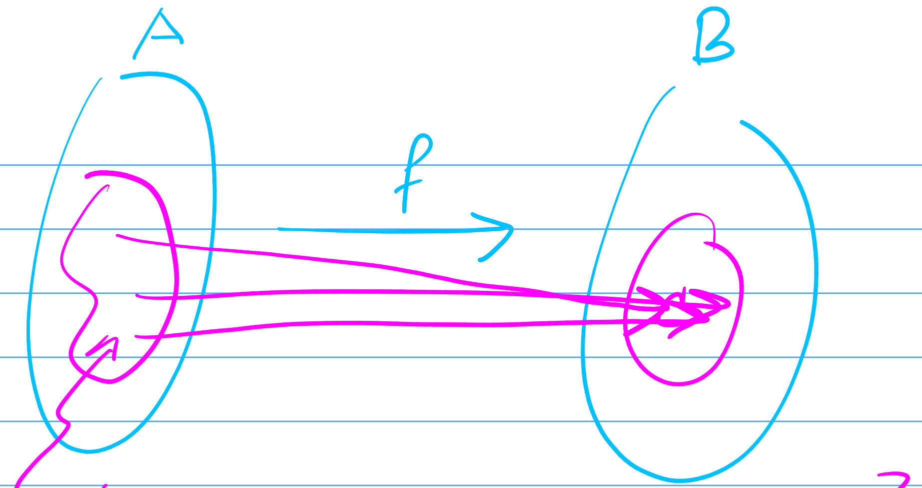
$X^{-1}(\{1\})$

More generally: $A \subset \mathbb{R}$

$$P(X \in A) = P(\{s \in S : X(s) \in A\})$$

$$= P(X^{-1}(A))$$

Inverse Image:



$f^{-1}(C) = \{a \in A : f(a) \in C\}$

Defn: Support of a RV (for now)

The set of possible values of X

(range)

Ex. (continue prev.)

$\text{Support}(X) = \{0, 1, 2, 3\}.$

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note: $P(X=5) = 0.$

More generally: If $A \cap \text{Support}(X) = \emptyset$

then $P(X \in A) = 0.$

Defn: Discrete and Continuous RVs

1) discrete RVs: support is finite or countable

Ex. X = sum of two dice

Ex. X = num. of customers arriving

2) Continuous RVs: support isn't countable

Ex. time/space

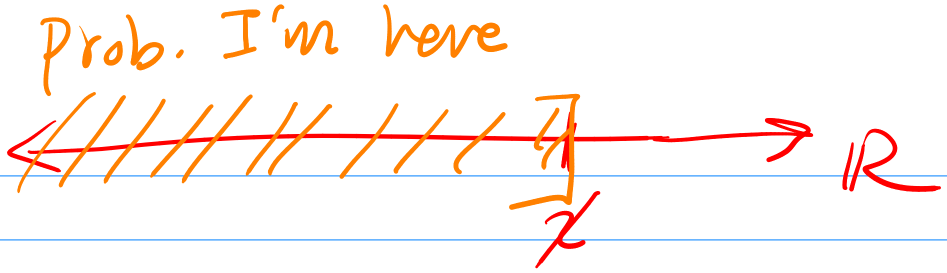
Defn: Cumulative Distribution Function (CDF)

If X is a RV then its CDF is a function

$$F: \mathbb{R} \to \mathbb{R}$$

defined for $x \in \mathbb{R}$ then

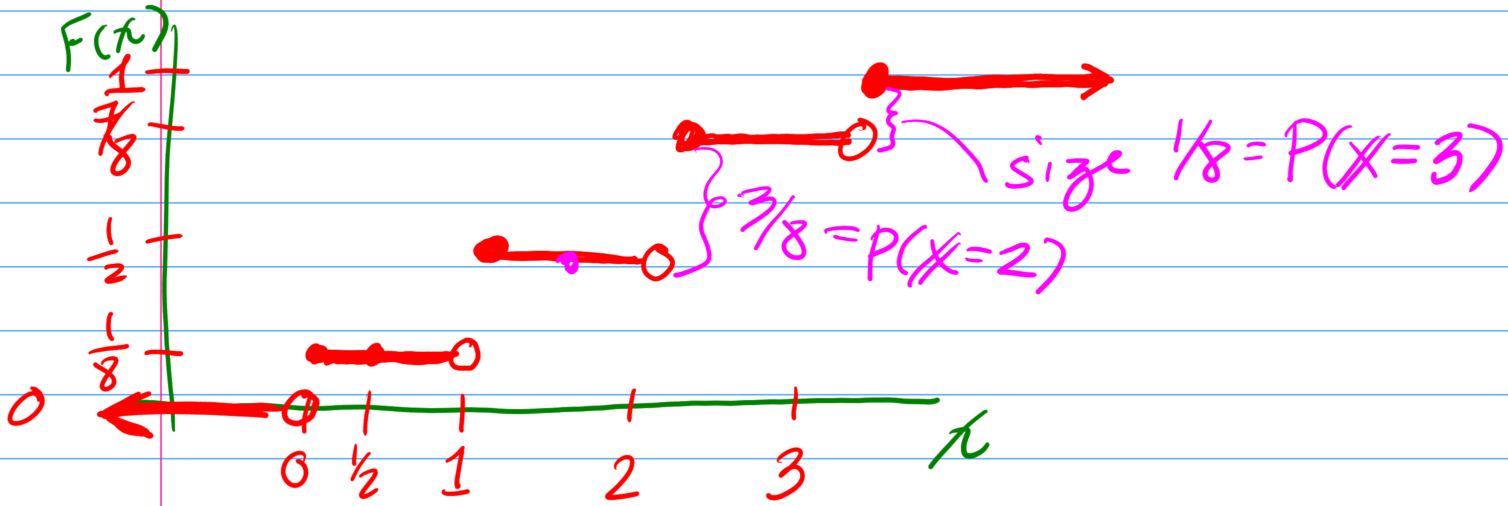
$$F(x) = P(X \le x).$$



$$P(X \le x) = P(X \in (-\infty, x])$$
$$= P(X^{-1}((-\infty, x])).$$

Ex. Toss coin 3 times.

X = # heads. Get $F(x)$.



$$F(0) = P(X \le 0) = P(X = 0) = 1/8$$
$$F(\tfrac{1}{2}) = P(X \le \tfrac{1}{2}) = P(X = 0) = 1/8$$
$$F(1) = P(X \le 1) = P(X = 1) + P(X = 0) = \frac{4}{8} = \frac{1}{2}$$
$$F(1.5) = P(X \le 1.5) = P(X \le 1) = 1/2$$
$$F(2) = P(X \le 2) = 7/8$$

$F(3) = P(X \le 3) = 1$

$F(4) = P(X \le 4) = 1$

$F(-1) = P(X \le -1) = 0$

Note: (1) this is a step fn

(2) steps occur at vals in support

(3) jump/step size is prob. of taking on that value

Facts:

(1) $\boxed{0 \le F(x) \le 1}$

Pf: $F(x) = P(\cdots\cdots) \in [0, 1]$.

(2) $\boxed{\lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} F(x) = 1.}$

(3) $\boxed{F \text{ is non-decreasing.}}$

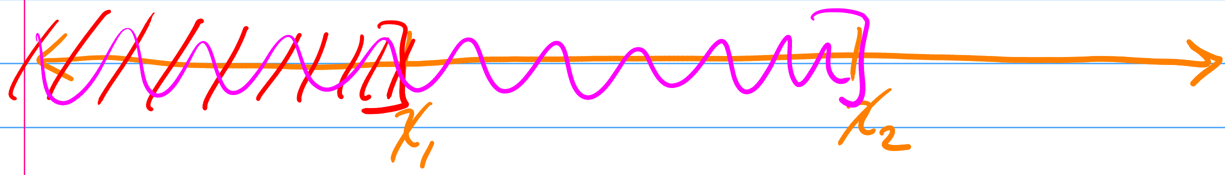
If $x_1 < x_2$ then $F(x_1) \le F(x_2)$.

$$F(x_1) \qquad \le \qquad F(x_2) =$$

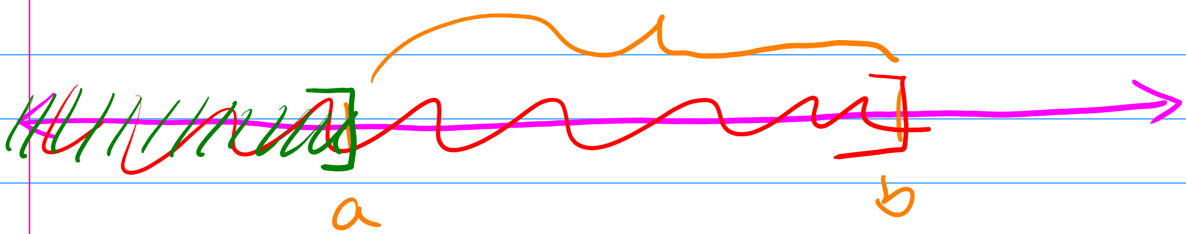
$$= P(X \le x_1)$$

$$= P(X \in (-\infty, x_1])$$

$$= P(\underbrace{X^{-1}((-\infty, x_1])}) \quad \le \quad = P(\underbrace{X^{-1}((-\infty, x_2])})$$

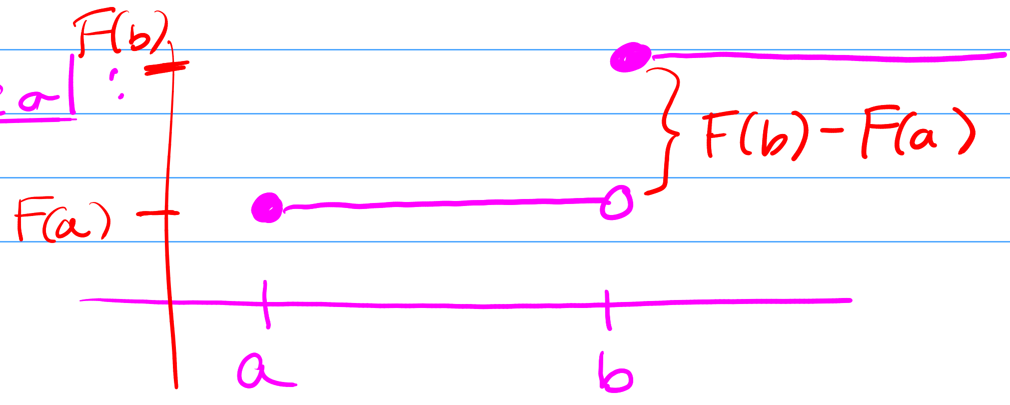


④ $$\boxed{\begin{array}{l} a \le b \ \text{ then} \\ \qquad P(a < X \le b) = F(b) - F(a) \end{array}}$$



$$(-\infty, b] \setminus (-\infty, a] = (a, b]$$

practical:



$$P(a < X \le b) = F(b) - F(a)$$
$$= \text{jump size at } x = b$$

$\parallel$

$P(X = b)$

⑤ $\boxed{F \text{ is right continuous}}$

<u>recall</u>: cts fn $\lim_{x \to a} F(x) = F(a)$.

<u>right cts</u>: $\lim_{x \to a^+} F(x) = F(a)$.

<u>Note</u>: cts fns <u>are</u> right cts.

<u>Theorem</u>:

F is the CDF of some RV

<u>iff</u>

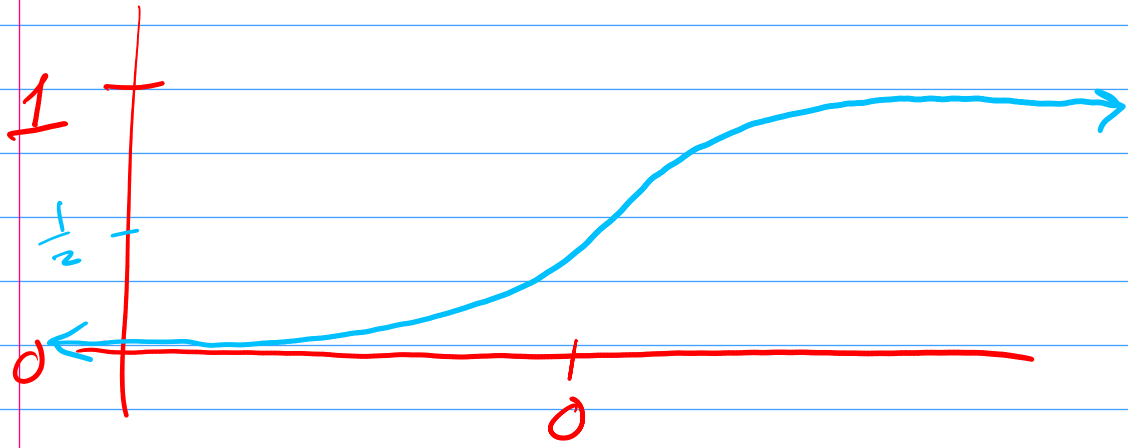
(1) $\lim_{x \to -\infty} F(x) = 0$ <u>and</u> $\lim_{x \to \infty} F(x) = 1$.

(2) F is non-decreasing

(3) F is right cts.

Ex, let

$$F(x) = \frac{1}{1+e^{-x}} \quad \text{for } x \in \mathbb{R}.$$



Q: is this a valid CDF?

Check 3 conditions:

① $\lim_{x \to -\infty} F(x) = \frac{1}{1+e^{-(-\infty)}} = \frac{1}{\infty} = 0$

$\lim_{x \to \infty} F(x) = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$

② $\frac{dF}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} > 0$

So F is increasing.

③ Right-continuous?

diff'able $\Rightarrow$ cts $\Rightarrow$ right cts.

Defn: Identical in Distribution

We say RVs X and Y are identical in dist. if

$\forall A \subset \mathbb{R}$ we have

$$P(X \in A) = P(Y \in A).$$

We write: $X \stackrel{d}{=} Y$.

This doesn't mean $X = Y$ (as functions)

Ex. 3 coin flips:

$X = \#$ heads

$Y = \#$ tails

there are diffent RVs.

$X(HHT) = 2$

$Y(HHT) = 1$

However, they have the same dist.

$$P(X=0) = \frac{1}{8} = P(Y=0)$$

$$P(X=1) = 3/8 = P(Y=1)$$

<u>Theorem:</u> $X \overset{d}{=} Y$

iff

$F_X = F_Y$ (as functions)

CDF of X.

<u>Ex.</u> Toss a coin (indep) until a H appears.

$$S = \{H, TH, TTH, TTTH, \ldots\}$$

Let p be the prob. of H on any flip.

X = # Flips until I get Hs

$s \in S$	$X(s)$
H	1
TH	2
TTH	3
TTTH	4
⋮	

Q: What's the CDF of X?

$$F(x) = P(X \le x)$$

Consider: $P(X = x)$

Let H_i = get H on i^{th} flip

$T_i = H_i^c$ = get T on i^{th} flip

$$P(X = i) = P(T_1 T_2 T_3 \cdots T_{i-1} H_i)$$
$$= P(T_1) P(T_2) \cdots P(T_{i-1}) P(H_i)$$
$$= (1-p)(1-p) \cdots (1-p) p$$
$$= (1-p)^{i-1} p.$$