Lecture 8

Ex, cont. from last time

$$F(x) = P(X \le x)$$

$$= P(X=1) + P(X=2) + P(X=3) + \cdots + P(X=x)$$

$$= \sum_{i=1}^{x} P(X=i)$$

$$= \sum_{i=1}^{x} (1-p)^{i-1} p$$

Geometric Sum:

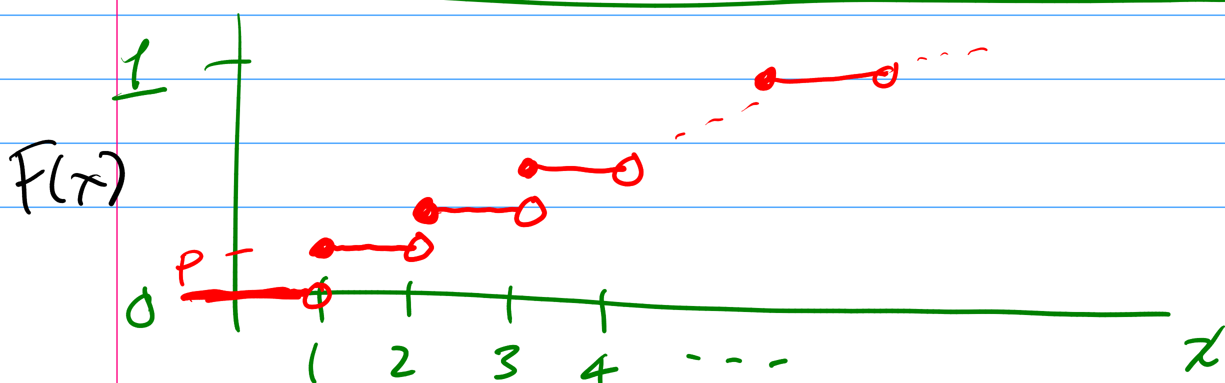
$$\underbrace{\sum_{i=0}^{n-1} r^i}_{} = \frac{1-r^n}{1-r}$$

$$= p \sum_{i=0}^{x-1} (1-p)^i$$

$r = 1-p$

$$= \cancel{p} \frac{1-(1-p)^x}{\cancel{1-(1-p)}}$$

$$\boxed{F(x) = 1-(1-p)^x \quad \text{for } x = 1, 2, 3, 4, \ldots}$$



$$F(x) = \begin{cases} 0, & x < 1 \\ 1-(1-p)^{\lfloor x \rfloor}, & x \geq 1 \end{cases}$$

$\lfloor x \rfloor = \text{floor}(x)$

Defn: discrete/cts

A discrete RV is one whose CDF is a step function.

A continuous RV is one whose CDF is continuous.

Defn: Probability Mass Function (PMF)

For a discrete RV X the PMF is a function

$$f : \mathbb{R} \to \mathbb{R}$$

So that $\forall x \in \mathbb{R}$

$$f(x) = P(X = x)$$

<u>Theorem:</u> For discrete RVs

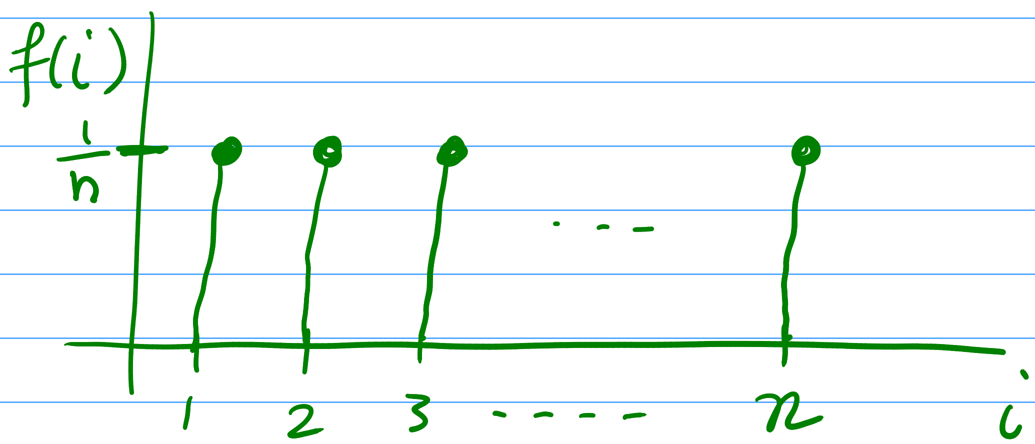
$$F(x) = \sum_{i \le x} f(i).$$

<u>Ex.</u> We say a RV X has a discrete Uniform dist over $1, \ldots, n$

$\boxed{\underline{\text{notation}}:\quad X \sim \mathcal{U}(\{1, \ldots, n\})}$

read: "distributed as"

if the PMF has the form



algebraically:

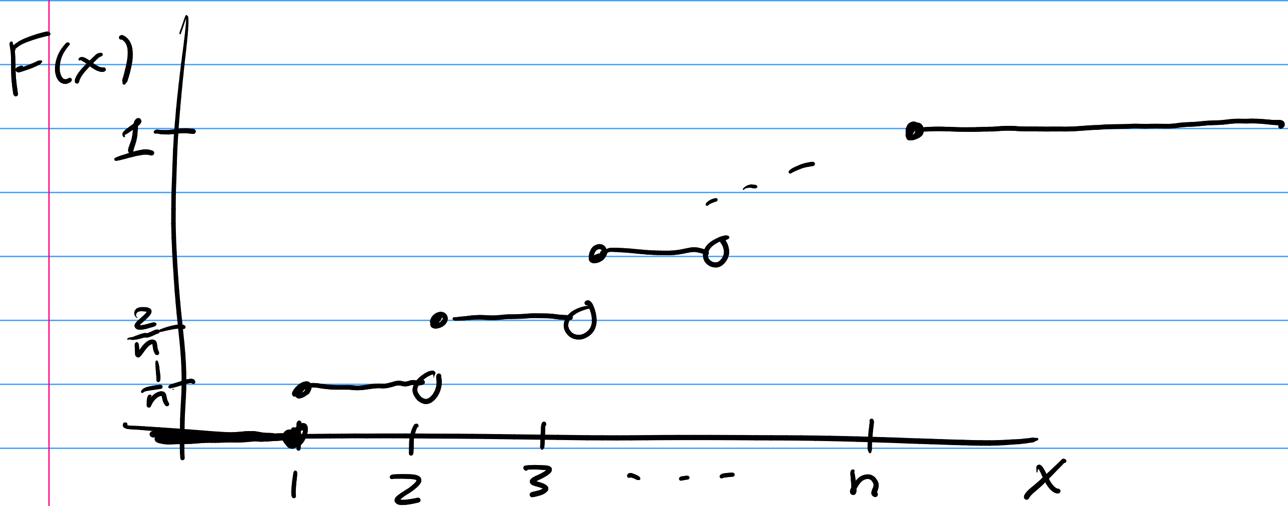
$$f(i) = \begin{cases} \frac{1}{n}, & x = 1, 2, \ldots, n \\ 0, & \text{else.} \end{cases}$$

Q: what's the CDF?

$$F(x) = \sum_{i \le x} f(i) = \sum_{i=1}^{x} \frac{1}{n} = \frac{x}{n}$$

for $x = 1, 2, \ldots, n$

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{\lfloor x \rfloor}{n}, & 1 \le x \le n \\ 1, & x > n \end{cases}$$



More generally, (for discrete)

$$P(X \in A) = \sum_{i \in A} f(i).$$

Ex. (continue) $X \sim U(\{1, \dots, 7\})$

$$P(2 \le X \le 5)$$

$$= P(X \in \{2,3,4,5\})$$

$$= \sum_{x=2}^{5} f(x)$$

$$= \sum_{x=2}^{5} 1/7 = 4/7.$$

Ex. Roll a die 60 times (indeply)

X = # 6s among rolls

What's the PMF of X?

$$f(0) = P(X=0) = \underbrace{(5/6)(5/6)(5/6)\cdots(5/6)}_{60 \text{ times}}$$

$$= (5/6)^{60}.$$

$$f(1) = P(X=1) = \binom{60}{1}\left(\tfrac{1}{6}\right)\underbrace{\left(\tfrac{5}{6}\right)\cdots\left(\tfrac{5}{6}\right)}_{59 \text{ rolls}}$$

$$= \binom{60}{1}\left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right)^{59}$$

$$\_\ \_\ \underline{6}\ \_\ \_\ \_\ \_\ \cdots\ \_\ \_\ \_$$

$$\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{1}{6}\right)\ \_\ \_\ \_\ \_$$

$$f(2) = P(X=2) = \binom{60}{2}\left(\tfrac{1}{6}\right)\left(\tfrac{1}{6}\right)\underbrace{\left(\tfrac{5}{6}\right)\cdots\left(\tfrac{5}{6}\right)}_{58}$$

$$\_\ \_\ \_\ \_\ \_\ \_\ \cdots\ \_\ \_\ \_$$

$$= \binom{60}{2}\left(\tfrac{1}{6}\right)^{2}\left(\tfrac{5}{6}\right)^{58}$$

General pattern:

$$\boxed{f(x) = \binom{60}{x}\left(\tfrac{1}{6}\right)^{x}\left(\tfrac{5}{6}\right)^{60-x}}$$

Called a Binomial RV:

I do a series of n independent experiments — each w/ a binary outcome (0 or 1) — and the prob. of a 1 is $p \in [0,1]$

Let X = # of 1s among n experiments.

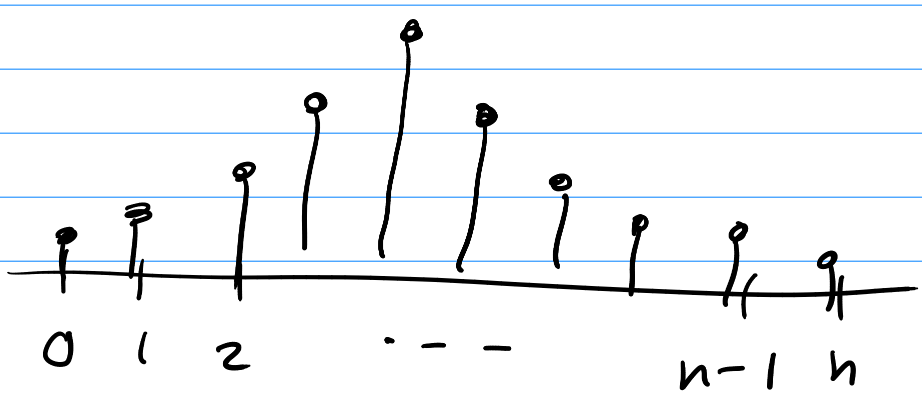
Then X has a binomial dist:

<u>notation:</u> $X \sim Bin(n, p)$.

<u>PMF:</u>

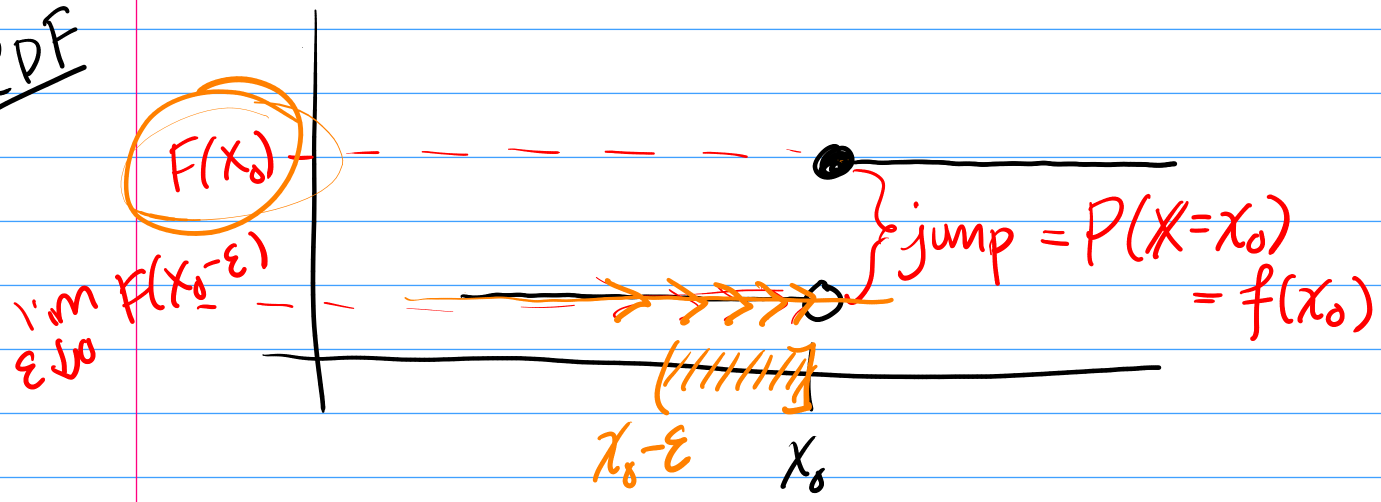
$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 0, 1, 2, \ldots, n$.



Discrete:

CDF



rule: $P(a < X \leq b) = F(b) - F(a)$

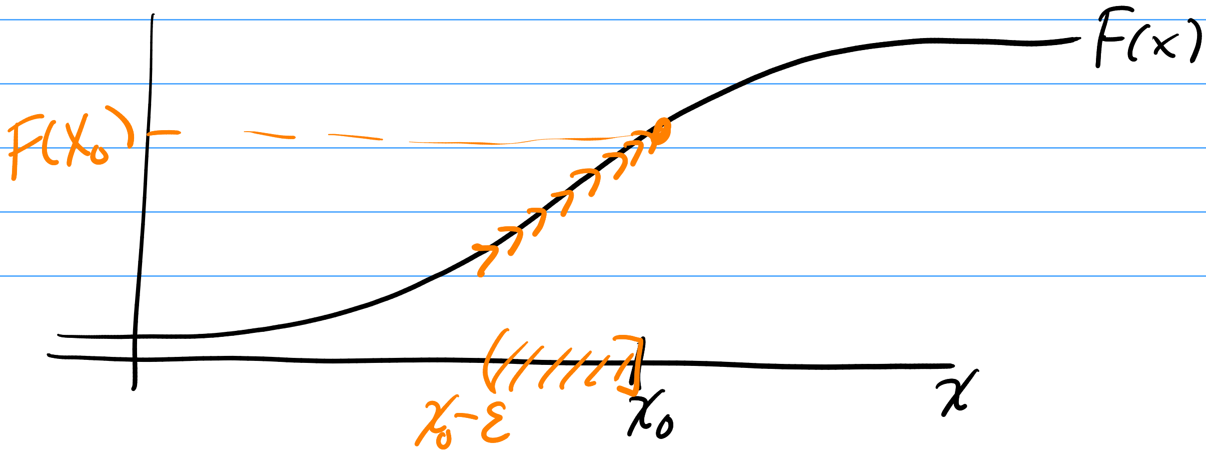
$$\lim_{\varepsilon \downarrow 0} P(x_0 - \varepsilon < X \leq x_0) = \lim_{\varepsilon \downarrow 0} F(x_0) - F(x_0 - \varepsilon)$$

$$= F(x_0) - \lim_{\varepsilon \downarrow 0} F(x_0 - \varepsilon)$$

$$\| $$

$$P(X = x_0) = \text{jump size}$$

what abt a cts RV?



$$P(X=x_0) = \cdots = F(x_0) - \lim_{\varepsilon \downarrow 0} F(x_0 - \varepsilon)$$

$$= F(x_0) - F(x_0)$$

$$= 0$$

Want is analogy for PMF:

$$F(x) = \sum_{i \le x} f(i) \quad . \quad (*)$$

Defn: Probability Density Function (PDF).

(cts ver of a PMF)

The PDF of a cts RV is a function

$$f : \mathbb{R} \to \mathbb{R}$$

defined so that for $x \in \mathbb{R}$

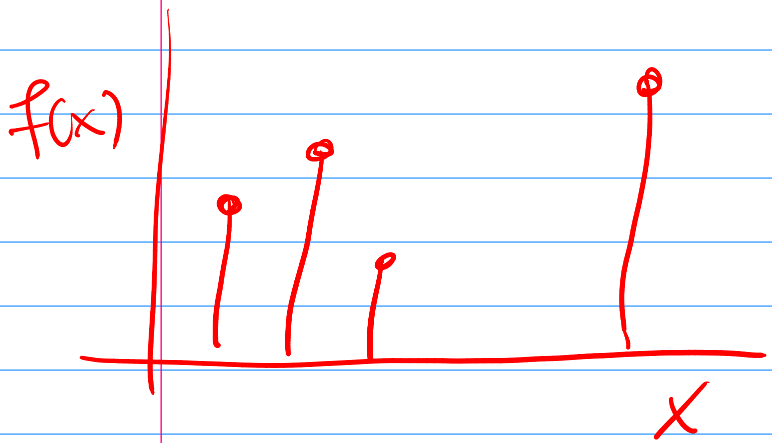
$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

By Fund. Thm of Calc:

$$\frac{dF}{dx} = \frac{d}{dx}\int_{-\infty}^{x} f(t)\,dt = f(x).$$

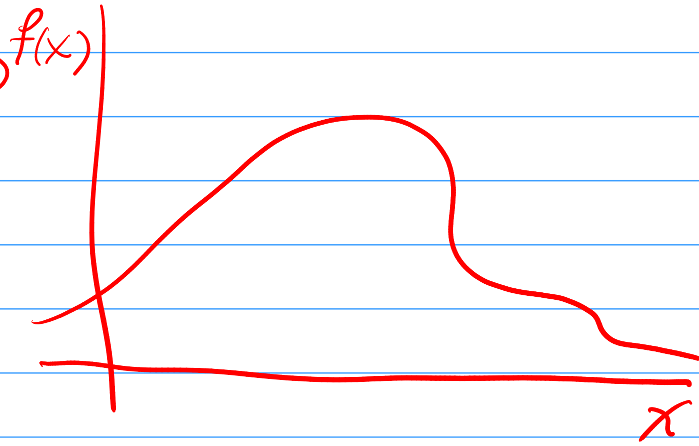
<u>Key:</u> PDF = deriv. of CDF.

discrete PMF



$f(x) = P(X=x)$

continuous PDF



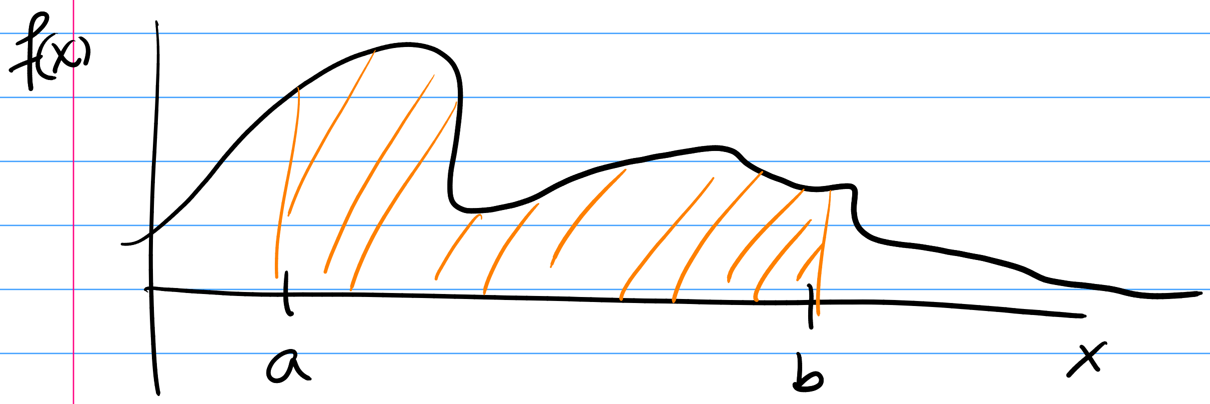
$f(x) \neq P(X=x)$.

<u>Properties of PDFs</u>

$$P(a < X \leq b) = F(b) - F(a)$$

$$= \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$

$$= \int_a^b f(t)\,dt$$



<u>Note:</u> $P(X=a) = 0 = P(X=b)$

so, $P(a<X\le b) = P(a \le X \le b)$
$= P(a \le X < b)$
$= P(a < X < b).$ $\Bigg\}$ <u>cts</u>

<u>Generally:</u>

(discrete) $P(X \in A) = \sum_{i \in A} f(i)$

(cts) $P(X \in A) = \int_A f(t)\,dt$