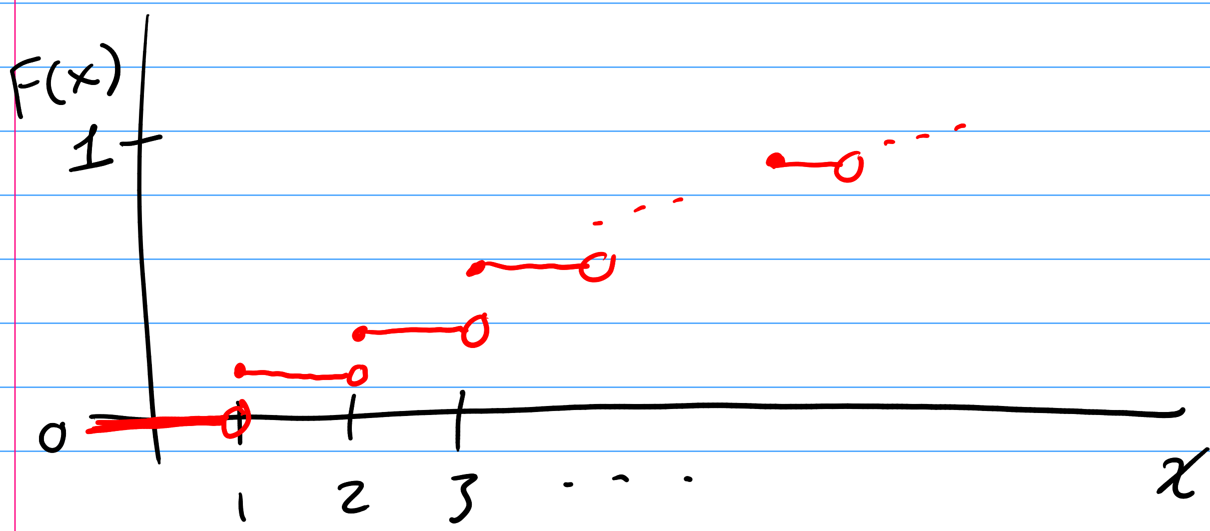
Lecture 8

$$\rightarrow F(x) = P(X \le x)$$
$$= P(X=1) + P(X=2) + \cdots + P(X=x)$$
$$\boxed{= \sum_{i=1}^{x} P(X=i)}$$
$$= \sum_{i=1}^{x} (1-p)^{i-1} p$$
$$= p \sum_{i=0}^{x-1} (1-p)^{i}$$
$$= \cancel{p} \frac{1-(1-p)^x}{\cancel{1-(1-p)}}$$

Geometric Sums:

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}.$$

$$\boxed{F(x) = 1-(1-p)^x \quad \text{for } x = 1, 2, 3, \ldots} \quad (*)$$



$$F(x) = \begin{cases} 0, & x < 1 \\ 1-(1-p)^{\lfloor x \rfloor}, & x \geq 1 \end{cases}$$

$\lfloor x \rfloor = \text{floor}(x)$

$= \text{rond-dawn}$

Defn: Discrete/Cts RVs

A <u>discrete RV</u> is one whose CDF is a step fn.

A <u>continuous RV</u> is one whose CDF is continuous.

Defn: Probability Mass Function (PMF)

For discrete RVs.

The PMF is a function

$$f : \mathbb{R} \to \mathbb{R}$$

So that for $x \in \mathbb{R}$

$$f(x) = P(X = x).$$

Theorem: For a discrete RV X,

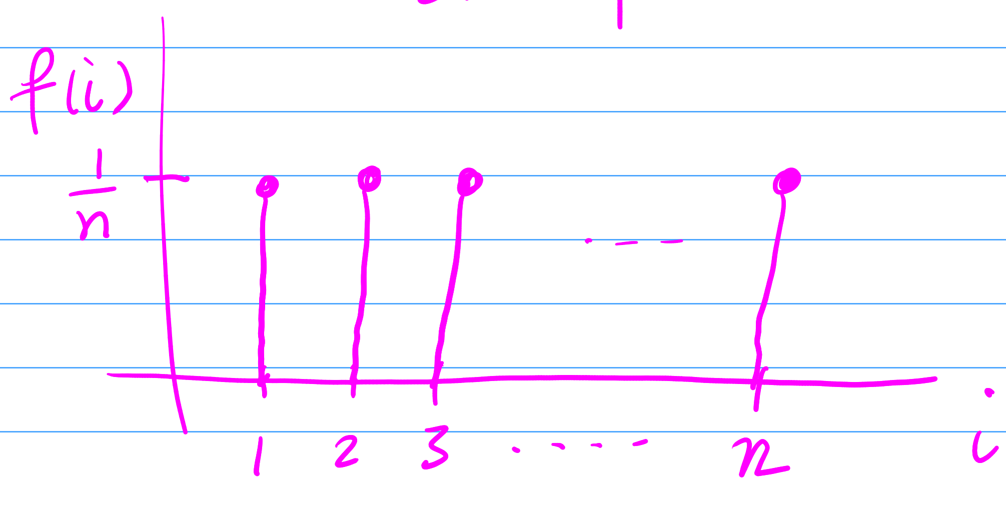
$$F(x) = \sum_{i \le x} f(i).$$

Ex. We say X has a discrete uniform dist over $1, \dots, n$

notation: $X \sim U(\{1, \dots, n\})$

"distributed as"

means: "stick plot"



algebraically:

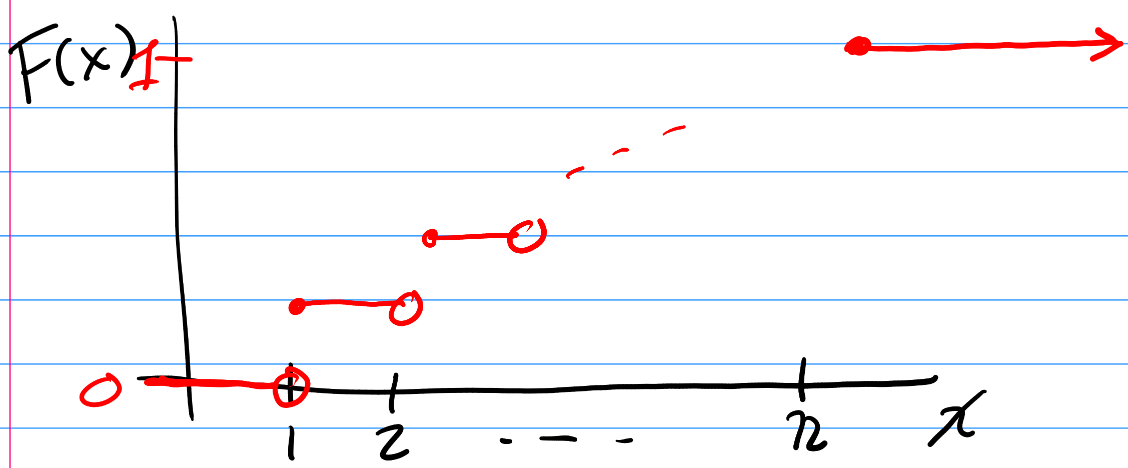
$$f(x) = \begin{cases} \frac{1}{n}, & x = 1, 2, \ldots, n \\ 0, & \text{else.} \end{cases}$$

Q: What's the CDF?

$$F(x) = \sum_{i \le x} f(i) = \sum_{i=1}^{x} \frac{1}{n} = \frac{x}{n}$$

all vals in support $\le x$.

for $x = 1, 2, 3, \ldots, n$



$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{\lfloor x \rfloor}{n}, & 1 \le x \le n \\ 1, & x > n \end{cases}$$

More generally, (discrete)

$$P(X \in A) = \sum_{i \in A} f(i)$$

Ex. $X \sim U(\{1, \dots, 7\})$

$$\boxed{f(x) = 1/7 \text{ for } x = 1, \dots, 7}$$

$$P(2 \le X \le 5)$$
$$= P(X \in \{2, \dots, 5\})$$
$$= \sum_{i=2}^{5} f(i) = \sum_{i=2}^{5} 1/7 = 4/7.$$

Ex. Roll a die 60 times. (indep)

X = # of 6s

What's the PMF of X?

$$f(0) = P(X = 0) = \underbrace{\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right) \cdots \left(\tfrac{5}{6}\right)}_{60}$$

$$= (5/6)^{60}.$$

$$\boxed{f(1) = P(X=1) = \binom{60}{1}\left(\frac{1}{6}\right)(5/6)^{59}}$$

$$\underline{\ \ }\ \underline{6}\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \cdots\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }$$

$$(5/6)(1/6)(5/6)\cdots \qquad (5/6)$$

$$f(2) = P(X=2) = \binom{60}{2}\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\cdots(5/6)$$

$$\underline{\ \ }\ \underline{6}\ \underline{\ \ }\ \underline{\ \ }\ \underline{6}\ \cdots\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }$$

$$\binom{60}{2}\left(\frac{1}{6}\right)^2(5/6)^{58}.$$

$$\boxed{f(x) = \binom{60}{x}\left(\frac{1}{6}\right)^x(5/6)^{60-x}}$$

Generically called a <u>Binomial R.V.</u>

I have a series of n independent binary (0 or 1) experiments — each w/ prob. p of 1 —

Then let X = # of 1s

then X has a Binomial dist:

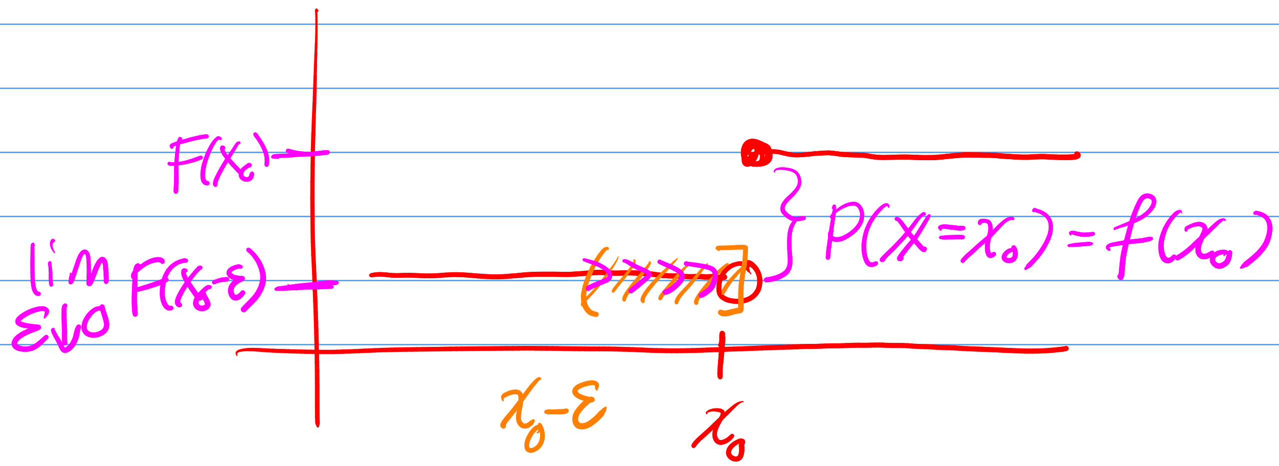
notation: $X \sim Bin(n, p)$

meaning:

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 0, 1, 2, \dots, n$

Discrete



$\underline{\text{rule}}$: $P(a < X \leq b) = F(b) - F(a)$

$$\lim_{\varepsilon \downarrow 0} P(x_0 - \varepsilon < X \leq x_0) = \lim_{\varepsilon \downarrow 0} F(x_0) - F(x_0 - \varepsilon)$$

$$\parallel$$

$$P(X = x_0) \qquad = F(x_0) - \lim_{\varepsilon \downarrow 0} F(x_0 - \varepsilon)$$

$$\parallel$$

$$f(x_0) \qquad = \text{jump size}$$

Continuous

$$\underline{P(X = x_0)} = F(x_0) - \lim_{\varepsilon \downarrow 0} F(x_0 - \varepsilon)$$

$$= F(x_0) - F(x_0) = \underline{0}$$

Want cts analogy for PMF:

$$F(x) = \sum_{i \leq x} f(i).$$

Defn: Probability Density Function (PDF)

Cts ver. of PMF

The PDF of a Cts RV is a function

$$f : \mathbb{R} \to \mathbb{R}$$

defined for $x \in \mathbb{R}$ as the function where

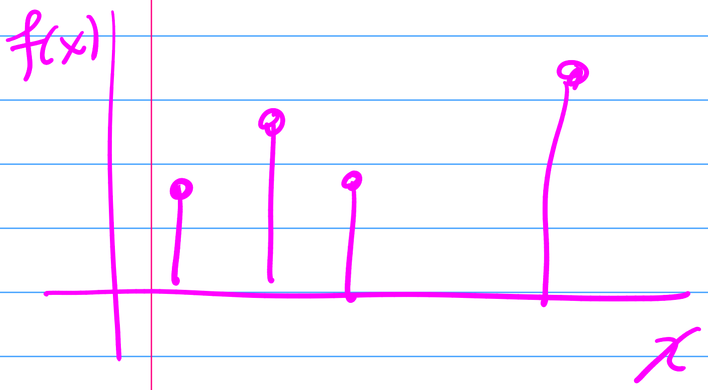
$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

Note by the fundamental thrm. of Calc.

$$\frac{dF}{dx} = \frac{d}{dx} \int_{-\infty}^{x} f(t)\,dt = f(x).$$

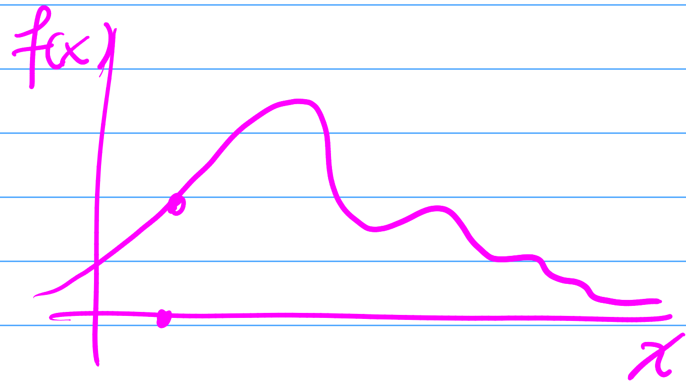
PDF = deriv. of CDF.

discrete PMF



$f(x) = P(X = x)$

Continuous PDF

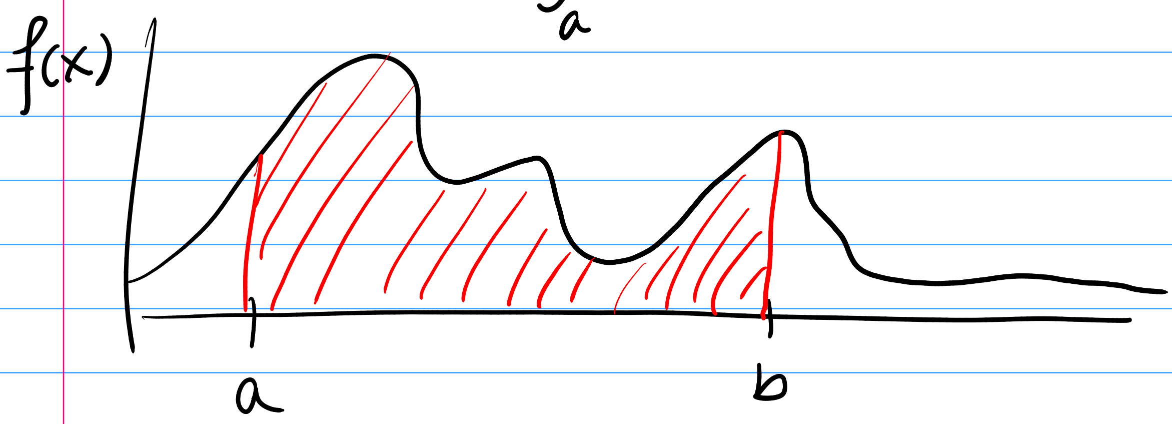


$f(x) \neq P(X = x)$

$$P(\underline{a < X \leq b}) = F(b) - F(a)$$

$$= \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt .$$

$$= \int_{a}^{b} f(t)\,dt$$



Nicely: $P(\mathbb{X}=a)=0$

$P(\mathbb{X}=b)=0$

$$P(a<\mathbb{X}\leq b)=P(a\leq \mathbb{X}\leq b)$$
$$=P(a<\mathbb{X}<b)$$
$$=P(a\leq \mathbb{X}<b).$$

Generally:

(discrete) $P(\mathbb{X}\in A)=\sum_{i\in A} f(i)$

(cts) $P(\mathbb{X}\in A)=\int_A f(t)\,dt.$
