Lecture 9:

Ex.

$$F(x) = \frac{1}{1+e^{-x}} \quad \text{for } x \in \mathbb{R}$$

Q: What's the PDF?

$$f(x) = \frac{dF}{dx} = \frac{d}{dx}\left[\frac{1}{1+e^{-x}}\right]$$

$$= \cdots$$

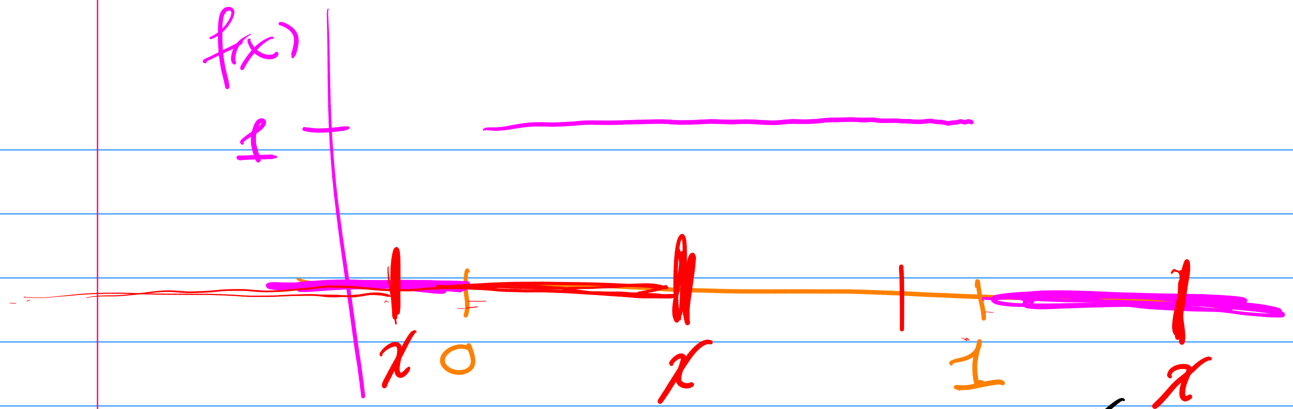
$$= \frac{e^{-x}}{(1+e^{-x})^2}$$

Ex. Continuous Uniform Dist

$$X \sim U(0,1)$$

means

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{else} \end{cases}$$



what's the CDF? $F(x) = \int_{-\infty}^{x} f(t)\,dt$

If $x<0$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

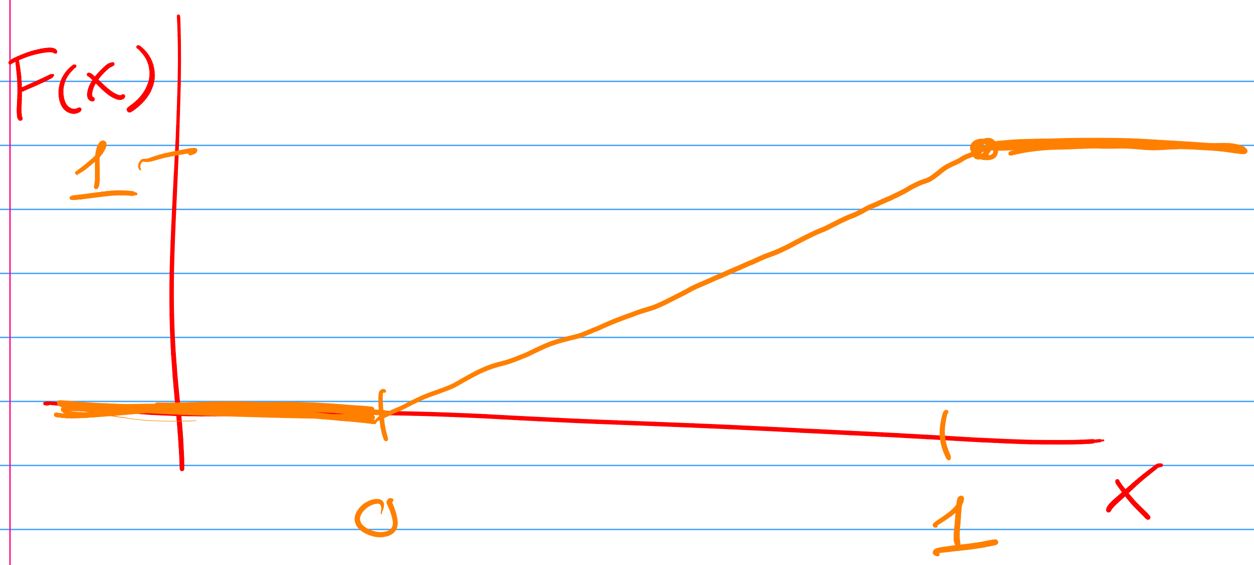
$0<x<1$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} 1\,dt = x$$

$x>1$

$$F(x) = \int_{0}^{1} 1\,dt = 1$$

$$F(x) = \begin{cases} 0, & x<0 \\ \boxed{x, \quad 0 \le x \le 1} \\ 1, & x>1 \end{cases}$$



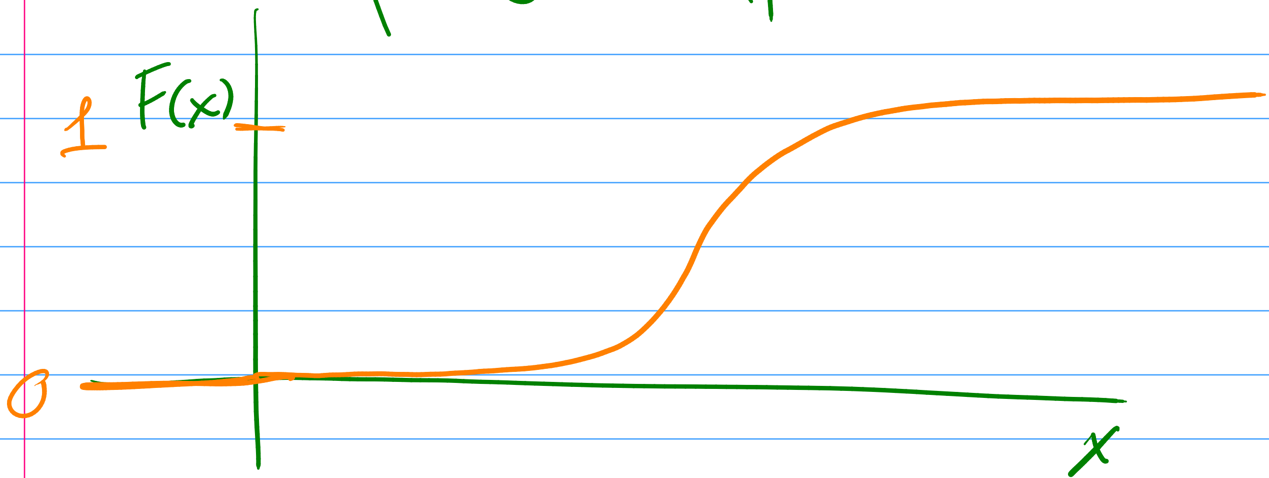
Ex.

$$f(x) = \begin{cases} x/2, & 0 < x < 2 \\ 0, & \text{else} \end{cases}$$

$$P(X > 1) = \int_1^\infty \underbrace{f(t)}_{\frac{x}{2}}\,dt = \underbrace{\tfrac{1}{2}\left(\tfrac{1}{2} + 1\right) \cdot 1}_{\text{area of trapezoid}}$$

$$= \frac{3}{4}$$

Ex $F(x) = \begin{cases} 1 - e^{-x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$



Q: $P(1 < X < 2)$?

Way 1: $P(1 < X < 2) = F(2) - F(1)$

$= (1 - e^{-2}) - (1 - e^{-1})$

$= e^{-1} - e^{-2}$

Way 2: $f(x) = \dfrac{dF}{dx} = e^{-x}$ for $x > 0$

$P(1 < X < 2) = \displaystyle\int_1^2 e^{-x}\,dx$

$= -e^{-x}\Big|_1^2 = e^{-1} - e^{-2}$

Theorem: PMF/PDF characterization

A function f is the PMF/PDF of some RV iff

① $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

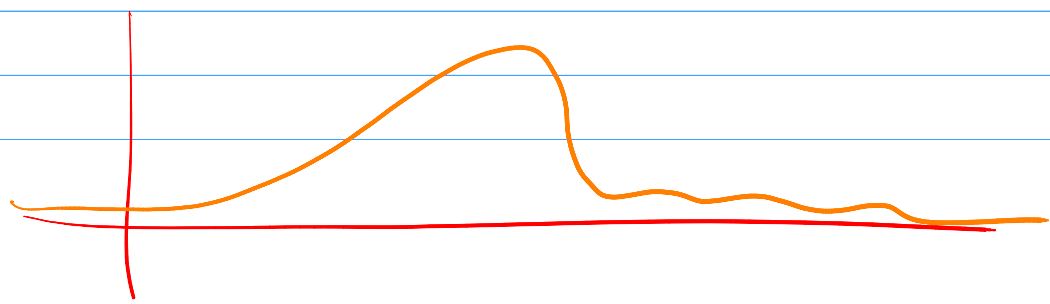
② (discrete) $\sum_{x \in \text{Support}} f(x) = 1$

(continuous) $\int_{\mathbb{R}} f(x)\,dx = 1$

If $g(x) \geq 0$ and $\int_{\mathbb{R}} g(x)\,dx = c < \infty$

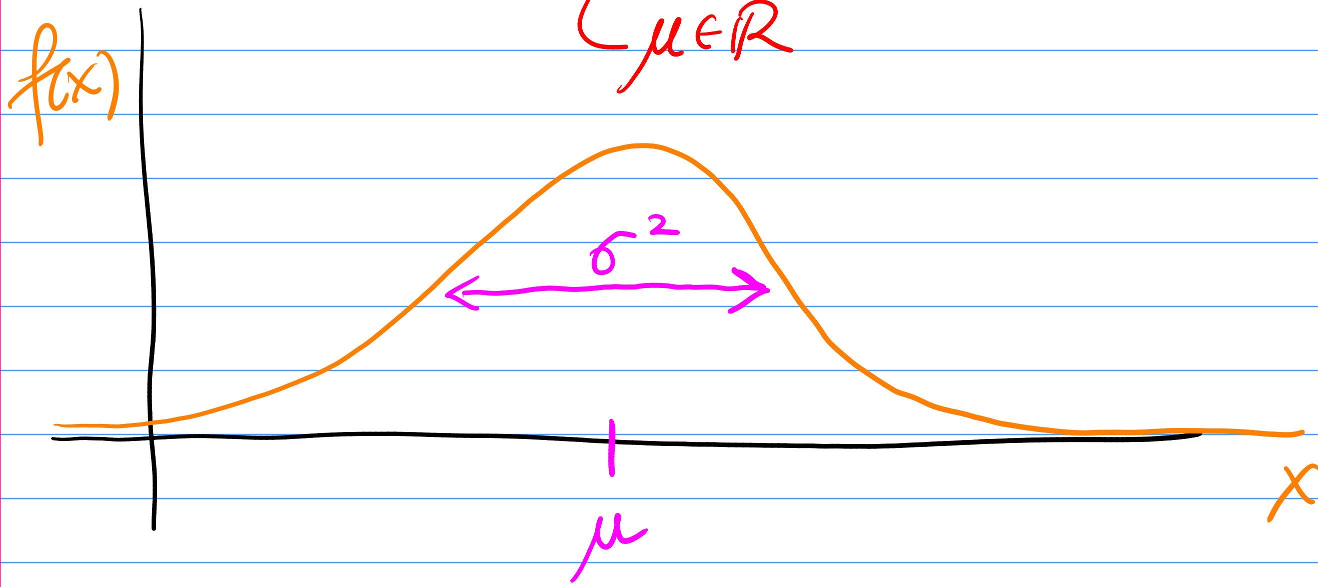
define $f(x) = \frac{1}{c} g(x)$

then f is a density function.



Ex. Normal Dist (Gaussian)

<u>notation</u>: $X \sim N(\mu, \sigma^2)$ — $\sigma^2 > 0$, $\mu \in \mathbb{R}$



$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

for $x \in \mathbb{R}$.

Defn: <u>Expected Value</u>

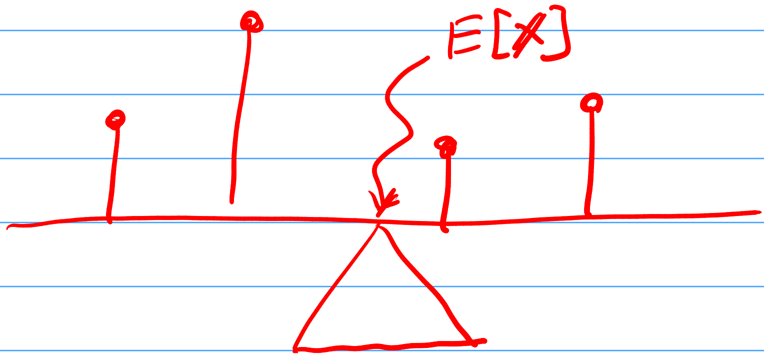
If X is a RV then the <u>mean</u> or <u>expected value</u> is denoted

$E[X]$

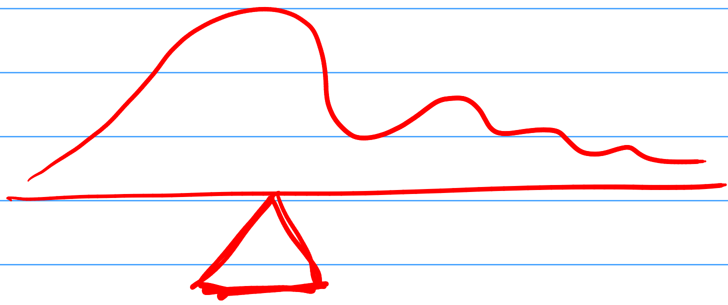
is defined as

PMF

(1) discrete $E[X] = \sum_{x} x f(x)$

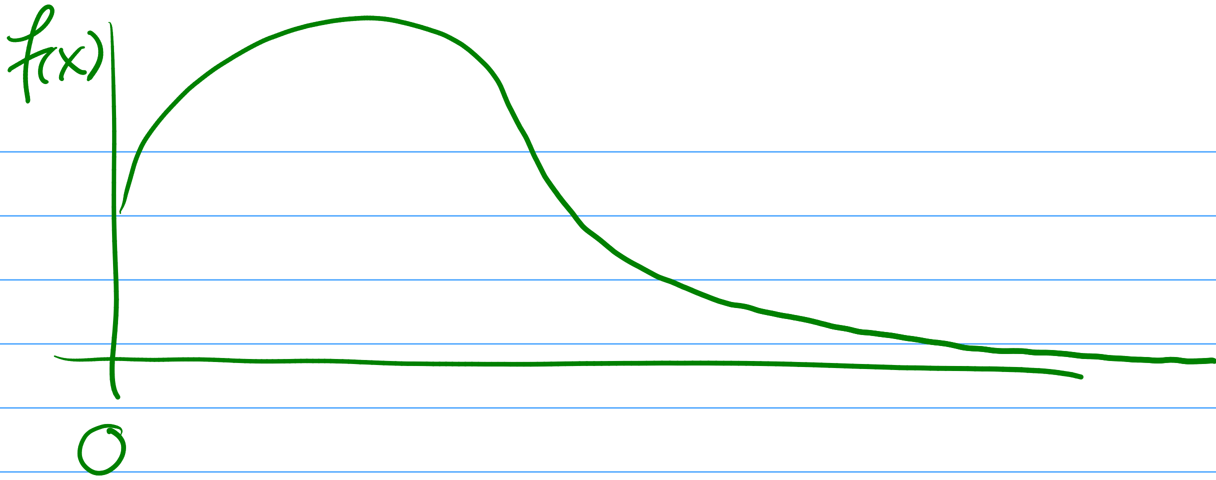


(2) cts: $E[X] = \int_{\mathbb{R}} x f(x)\,dx$



Ex. Let $X \sim \mathrm{Exp}(\lambda)$ rate $\lambda > 0$

"X has an exponential dist"



$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

Q: $E[X] = \int_{\mathbb{R}} x f(x)\, dx$

$$= \int_0^\infty x \lambda e^{-\lambda x}\, dx$$

by parts: $u = x$, $du = dx$; $v = -e^{-\lambda x}$, $dv = \lambda e^{-\lambda x}\, dx$

$$\int u\, dv = uv - \int v\, du$$

$$= \left[-x e^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}\, dx$$

$$= -[0 - 0] + \int_0^\infty e^{-\lambda x}\, dx$$

$$= \frac{1}{\lambda} \int_0^\infty \underbrace{\lambda e^{-\lambda x}}_{f(x)} dx$$

$$\underbrace{\qquad\qquad\qquad}_{1}$$

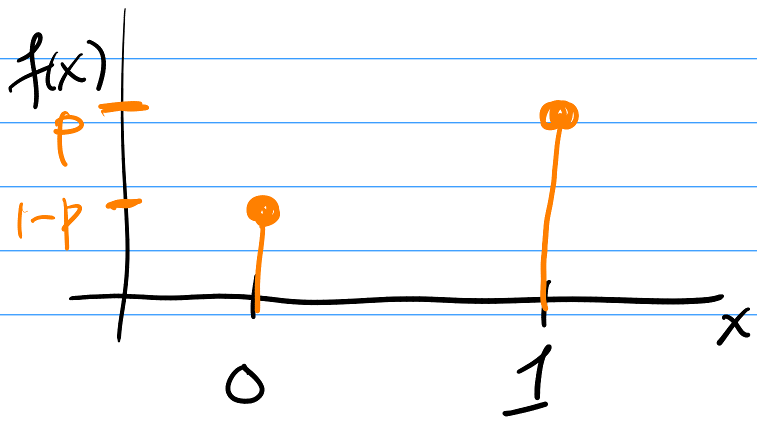
$$E[X] = \frac{1}{\lambda}$$

Ex. $X \sim \text{Bern}(p)$ $\quad p \in [0,1]$

bernoulli

X = any experiment w/ a 0/1 outcome w/ prob p of getting a 1.

$$f(x) = \begin{cases} 1-p, & x = 0 \\ p, & x = 1 \end{cases}$$



$$E[X] = \sum_{x=0,1} x f(x)$$

$$= (0)f(0) + (1)f(1)$$

$$= f(1) = p.$$

Binomial RV:

$$X \sim Bin(n, p)$$

X = do a series of n bernaulli experiments and cant how many '1's

= sum of n bernaulli RVs

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 0, 1, 2, \ldots, n$

$$E[X] = np.$$

Functions of RVs

A function of a RV is a RV:

e.g. If X is a RV then so is

$$X^2, \ \log X, \ \sqrt{X}, \ \dots \text{ etc.}$$

Theorem: Law of the Unconscious Statistician

If $g : \mathbb{R} \to \mathbb{R}$ and X is a RV

then

$$E[g(X)] = \begin{cases} \sum_x g(x) f(x) & \text{(discrete)} \\ \int_{\mathbb{R}} g(x) f(x)\,dx & \text{(cts)} \end{cases}$$

Ex. $X \sim \text{Exp}(\lambda)$

$$E[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx$$

$$= \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx$$

$$u = x^2 \qquad v = -e^{-\lambda x}$$

$$du = 2x\,dx \qquad dv = \lambda e^{-\lambda x}\, dx$$

$$= \int u\, dv = uv - \int v\, du$$

$$= (x^2)(-e^{-\lambda x}) \Big|_0^\infty + 2\int_0^\infty e^{-\lambda x} x\, dx$$

$$= (0 - 0) + \frac{2}{\lambda} \int_0^\infty \lambda e^{-\lambda x} x\, dx$$

$$= \frac{2}{\lambda} \int_0^\infty x \underbrace{\lambda e^{-\lambda x}}_{f(x)} dx$$

$$= \frac{2}{\lambda} E[X]$$

$$= \frac{2}{\lambda} \frac{1}{\lambda} = \frac{2}{\lambda^2} = E[X^2].$$