Lecture 9:

Ex. $F(x) = \dfrac{1}{1+e^{-x}}$ for $x \in \mathbb{R}$

Q: What's the PDF?

$$f(x) = \frac{dF}{dx} = \frac{d}{dx}\left(\frac{1}{1+e^{-x}}\right)$$

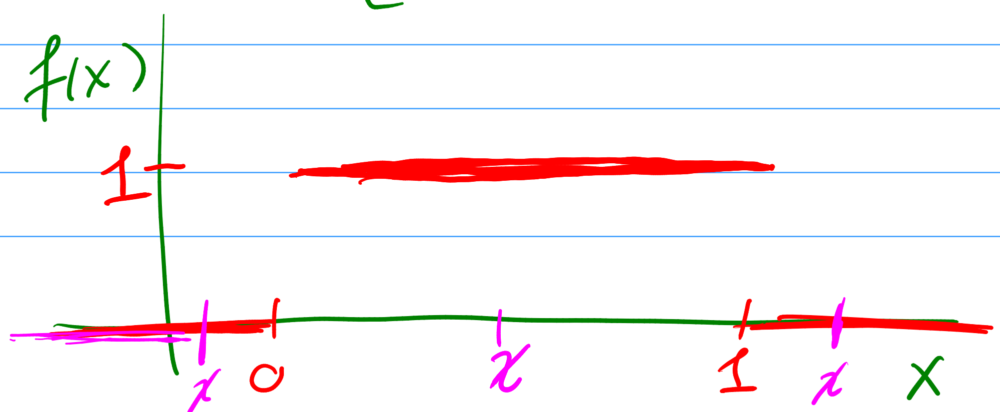
$$= \cdots = \frac{e^{-x}}{(1+e^{-x})^2}.$$

Ex. Continuous Uniform

$X \sim U(0,1)$

means

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{else} \end{cases}$$



what's the CDF?

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

For $x<0$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

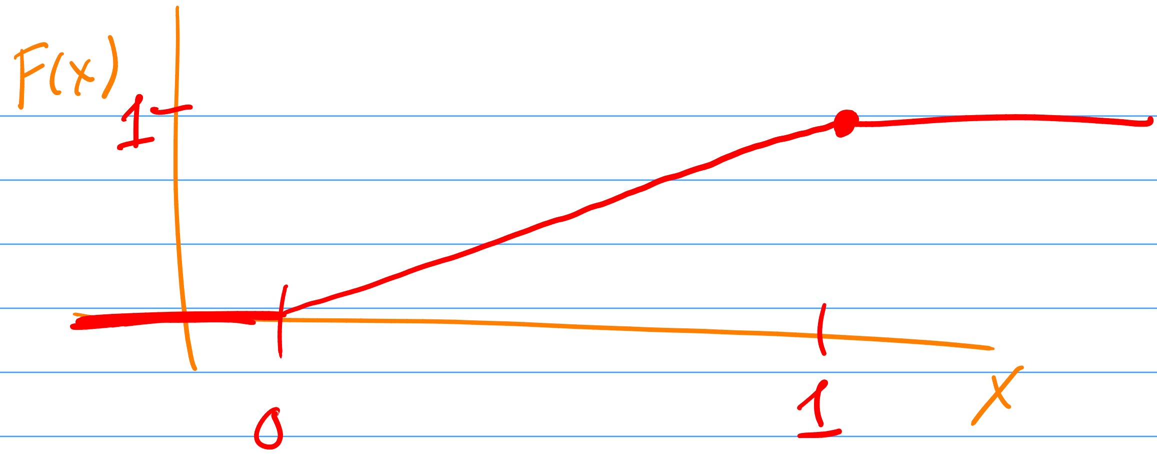
For $0<x<1$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} 1\,dt = x$$

For $x>1$

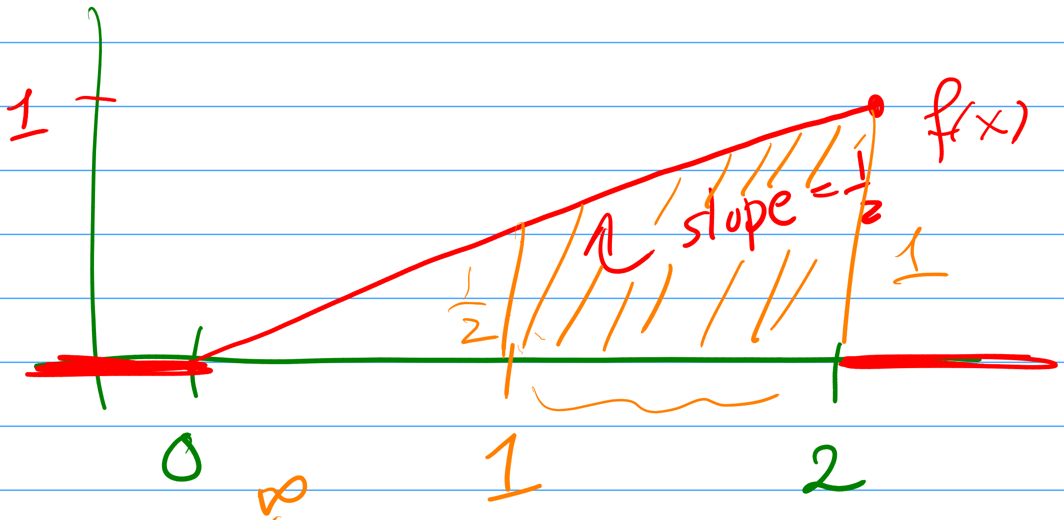
$$F(x) = \int_{0}^{1} 1\,dt = 1$$

$$F(x) = \begin{cases} 0, & x<0 \\ x, & 0\le x\le 1 \\ 1, & x>1 \end{cases}$$



Ex,

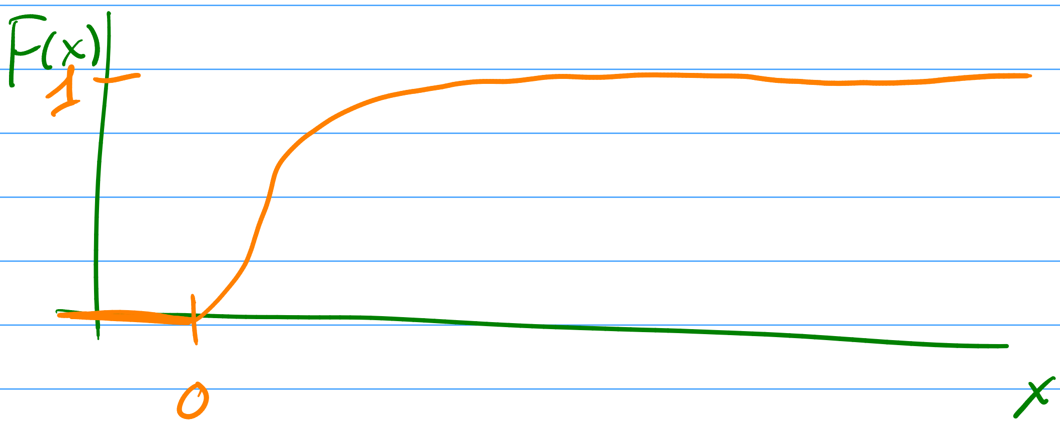
$$f(x) = \begin{cases} x/2, & 0 < x < 2 \\ 0, & \text{else} \end{cases}$$



$$P(X > 1) = \int_1^\infty f(x)\,dx$$

$$= \int_1^2 x/2\,dx = \frac{1}{2}\left(1 + \frac{1}{2}\right) \cdot 1 = \frac{3}{4}$$

Ex, $F(x) = 1 - e^{-x}$ for $x > 0$



$P(1 < X < 2)$

Way 1: $P(a < X \le b) = F(b) - F(a)$

$$P(1 < X < 2) = F(2) - F(1)$$
$$= (1 - e^{-2}) - (1 - e^{-1})$$
$$= e^{-1} - e^{-2}$$

Way 2:

$$f(x) = \frac{dF}{dx} = \frac{d}{dx}(1 - e^{-x}) = e^{-x} \text{ for } x > 0$$

$$P(1 < X < 2) = \int_1^2 f(x)\,dx$$
$$= \int_1^2 e^{-x}\,dx = e^{-1} - e^{-2}.$$

Theorem: PMF/PDF characterization

A function f is the PMF/PDF of some RV iff

1. $f(x) \geq 0 \quad \forall x \in \mathbb{R}$
2. (discrete) $\sum_x f(x) = 1$

   (cts) $\int_{\mathbb{R}} f(x)\,dx = 1$.

(cts) If $g(x) \geq 0$ and $\int_{\mathbb{R}} g(x)\,dx = c < \infty$

define $f(x) = \frac{1}{c} g(x)$

then f is a density.

PMF: $0 \leq f(x) \leq 1$

PDF: Can be that $f(x) > 1$.

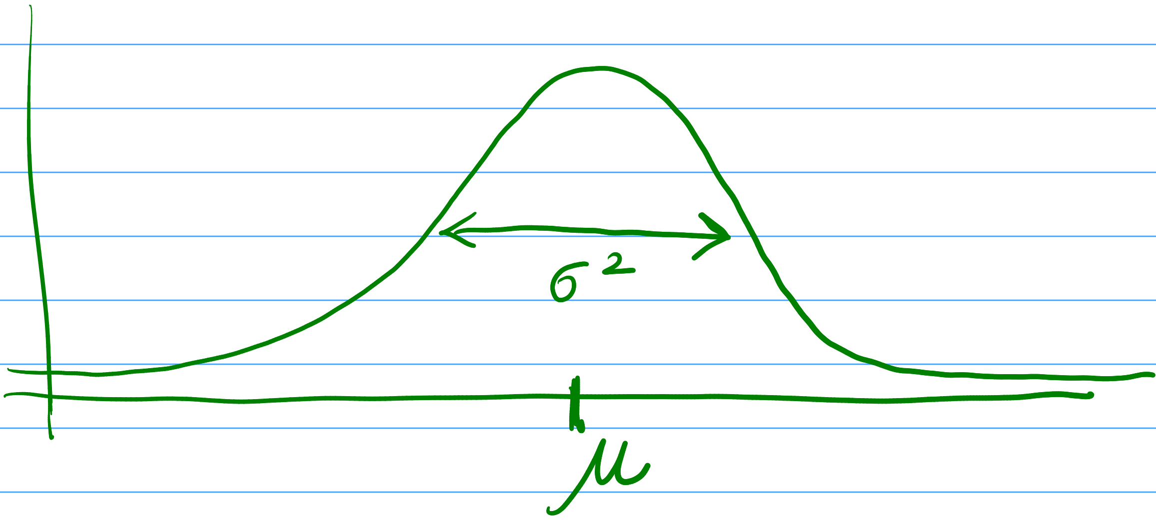
Must be that $f(x) \geq 0$.

Ex. Normal Distribution (Gaussian)

notation: $X \sim N(\mu, \sigma^2)$

mean $\mu \in \mathbb{R}$

variance $\sigma^2 > 0$



$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

for $x \in \mathbb{R}$

looks like: e^{-x^2}

Defn: Expected Value

The mean or expected value

of a RV X denoted

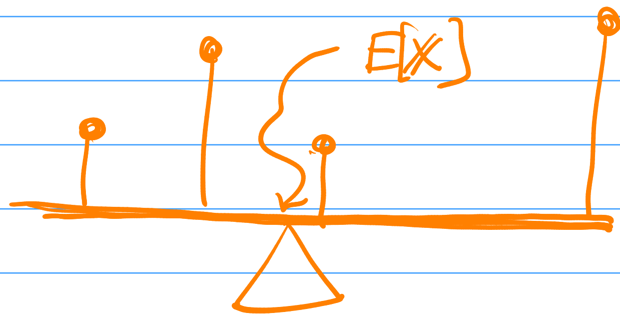
$$E[X]$$

is defined as

(1) <u>discrete:</u>

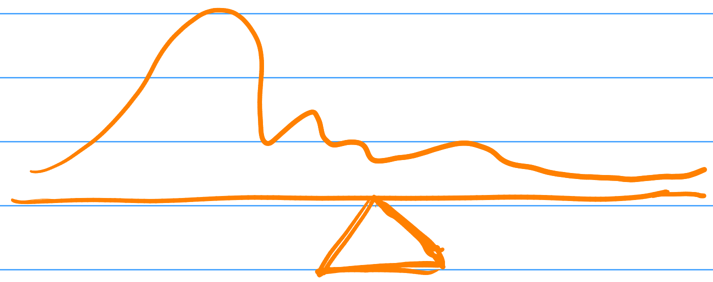
PMF

$$E[X] = \sum_x x f(x)$$



(2) <u>cts:</u>

$$E[X] = \int_{\mathbb{R}} x f(x) dx$$

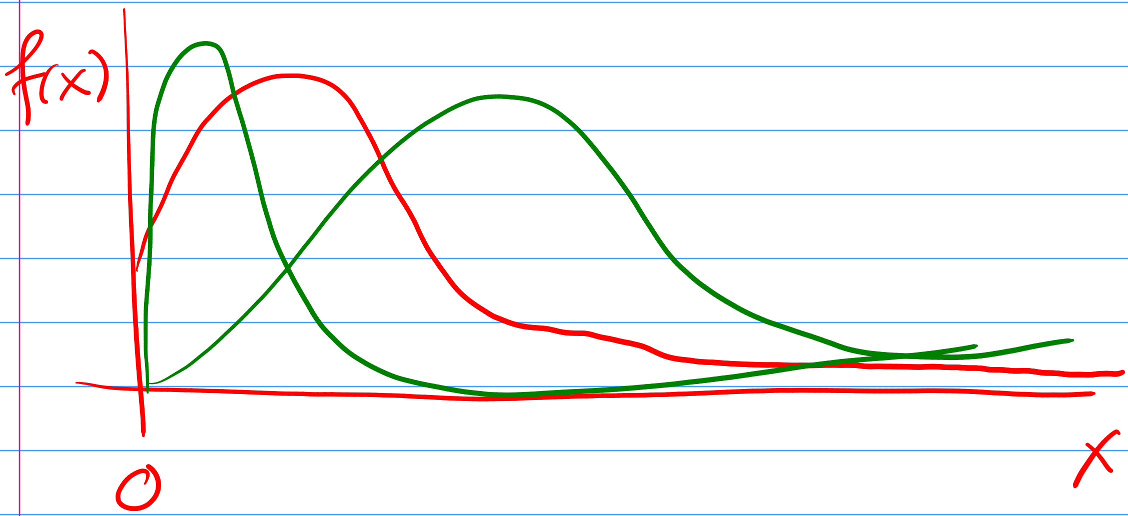


Exponential dist

Ex, $X \sim \text{Exp}(\lambda)$ "rate"

$\lambda > 0$

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0$$



$E[X]$?

$$E[X] = \int_{\mathbb{R}} x f(x)\, dx$$

$$= \int_0^\infty \underbrace{x}_{u} \underbrace{\lambda e^{-\lambda x} dx}_{dv}$$

by parts:

$u = x \qquad v = -e^{-\lambda x}$

$du = dx \qquad dv = \lambda e^{-\lambda x} dx$

$$\int u\,dv = uv - \int v\,du$$

$$= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\,dx$$

$$= (0-0) + \int_0^\infty e^{-\lambda x}\,dx$$

$$= -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty$$

$$= -\frac{1}{\lambda}(0-1)$$

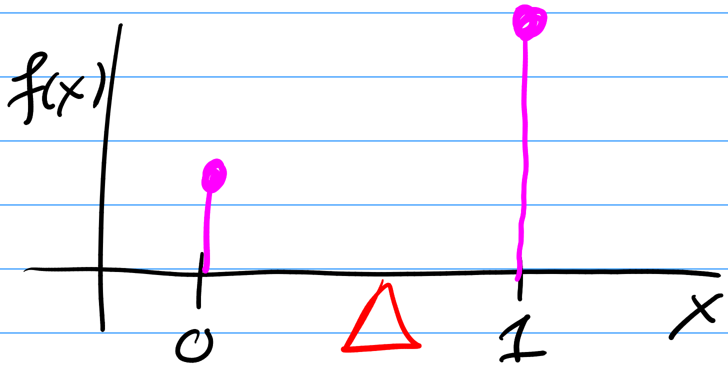
$$E[X] = 1/\lambda \,.$$

Ex. $X \sim \text{Bern}(p)$ $p \in [0,1]$

Bernoulli

X = any experiment w/ a 0/1 outcome where prob. of a 1 is p.

$$f(x)=\begin{cases} p, & x=1 \\ 1-p, & x=0 \end{cases}$$



$$E[X]=\sum_{x=0,1} x f(x)$$

$$=(0)f(0)+(1)f(1)$$

$$=f(1)$$

$$=p$$

<u>Binomial RV</u>:

$$X \sim \text{Bin}(\underbrace{n}_{\text{integer}}, \underbrace{p}_{[0,1]})$$

X = do n Bernalli trials and count the number of 1s

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$x = 0, 1, 2, \ldots, n.$$

$\text{Bin}(1, p) = \text{Bern}(p).$

$$E[X] = np.$$

Functions of RVs:

A function of a RV is a RV.

e.g. if X is a RV then so is

$$X^2, \log(X), \sqrt{X}, \ldots$$

Theorem: Law of the Unconscious Statistician

If $g: \mathbb{R} \to \mathbb{R}$ and X is a RV

$$E[g(X)] = \begin{cases} \sum_x g(x) f(x) & \text{(discrete)} \\ \int_{\mathbb{R}} g(x) f(x)\, dx & \text{(cts)} \end{cases}$$

Ex, $X \sim Exp(\lambda)$

$$E[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx$$

$$= \int_0^\infty x^2 \lambda e^{-\lambda x}\,dx$$

$u = x^2$ $\qquad$ $v = -e^{-\lambda x}$

$du = 2x\,dx$ $\qquad$ $dv = \lambda e^{-\lambda x}dx$

$$= uv - \int v\,du$$

$$= -x^2 e^{-\lambda x}\Big|_0^\infty + 2\int_0^\infty e^{-\lambda x} x\,dx$$

$$= 0 + 2\int_0^\infty x e^{-\lambda x}dx$$

$$= \frac{2}{\lambda}\int_0^\infty x \underbrace{\lambda e^{-\lambda x}}_{f(x)}dx$$

$$= \frac{2}{\lambda}\underbrace{\int_0^\infty x f(x)\,dx}_{E[X]} = \frac{2}{\lambda}\frac{1}{\lambda} = \frac{2}{\lambda^2}.$$