Review

SP 2.4 Show: $P(AB) \geq P(A) + P(B) - 1$

$$P(\underbrace{A \cup B}_{}) = P(A) + P(B) - P(AB)$$

$$P(E) \leq 1$$

So $P(A \cup B) \leq 1$

$$P(A) + P(B) - P(AB) \leq 1$$

$$\boxed{P(A) + P(B) - 1 \leq P(AB)}$$

SP 13.2

$$X_i \overset{iid}{\sim} N(\mu, \sigma^2)$$

Want: MGF of $Z = \sum_{i=1}^{p} X_i$

Bivariate: $X \perp Y$ then $M_{X+Y}(t) = M_X(t) M_Y(t)$

Multivariate: X_i indep $M_{\sum_{i=1}^{n} X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t)$

$$\boxed{M_Z(t) = \prod_{i=1}^{p} M_{X_i}(t)}$$

$$e^{(\mu t + \sigma^2 t^2/2)}$$

$$e^a e^b = e^{a+b}$$

$$\prod_{i=1}^{p} e^{a_i} = e^{\sum_{i=1}^{p} a_i}$$

$$= \prod_{i=1}^{p} e^{\mu t + \sigma^2 t^2/2}$$

$$= e^{(\mu p)t + (p\sigma^2)t^2/2}$$

MGF of $N(\mu p, p\sigma^2)$

$$\boxed{Z = \sum_{i=1}^{N} X_i \sim N(p\mu, p\sigma^2)}$$

$X \sim N(\mu, \sigma^2)$

$$\boxed{Y = pX \sim N(p\mu, p^2\sigma^2)}$$

SP 13.5 $\quad \underset{\sim}{X} \sim N(\mu, \Sigma)$

B s.t. $B\Sigma B^T = I$

$$Z = B(\underset{\sim}{X} - \mu) = \underbrace{-\mu B}_{a} + \underline{B}\underset{\sim}{X}$$

$$\tilde{z} = \boxed{a + \underline{B}\,\tilde{\cancel{X}} \sim N\big(\overset{0}{\overline{a + B\mu}},\ \overset{I}{\overline{B\Sigma B^T}}\big)}$$

$$a + B\mu = -\mu B + B\mu = 0$$

$$\boxed{z \sim N(0, I_p)}$$

$$I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}_{p \times p}$$

$$b\sigma^2 b = 1 \Rightarrow b^2 = 1/\sigma^2$$

$$b = 1/\sqrt{\sigma^2} = (\sigma^2)^{-1/2}$$

$$B\Sigma B^T = I \ , \quad B = \Sigma^{-1/2}$$

$$= (\Sigma^{-1})^{1/2}$$

$$= (\Sigma^{1/2})^{-1}$$

$$\boxed{\Sigma = AA} \Rightarrow A = \Sigma^{1/2}$$

$$B = A^{-1} = \Sigma^{-1/2}$$

$$Z = \Sigma^{-1/2}(\underset{\sim}{X} - \mu)$$

$$B\Sigma B^T = \Sigma^{-1/2}\,\Sigma\,\Sigma^{-1/2} = I$$

$$\Sigma = U\Lambda U^T$$

$$\Sigma^{1/2} = U\Lambda^{1/2}U^T$$

$(\mu_1, \mu_2, \mu_3, \mu_4)$

$$\underset{\sim}{X} \sim N(\mu, \Sigma)$$

$$\underset{\sim}{X} = (X_1, X_2, X_3, X_4)$$

$B\Sigma B^T$

$$Z = (X_1, X_3, X_4) \sim N\left((\mu_1, \mu_3, \mu_4), \begin{bmatrix} \Sigma_{11} & \Sigma_{13} & \Sigma_{14} \\ \Sigma_{31} & \Sigma_{33} & \Sigma_{34} \\ & & \Sigma_{44} \end{bmatrix}\right)$$

$$Z = B\underset{\sim}{X}$$

$B\Sigma B^T$

$$\underset{3\times4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \quad 4\times1$$

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + \sum_{i=1}^{n}\sum_{j\neq i} Cov(X_i, X_j)$$

$$+2\sum_{i=1}^{n}\sum_{j<i} \sim$$

$$\boxed{Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)}$$

$B\underset{\sim}{X} = \sum X_i \sim N(B\mu, \underbrace{B\Sigma B^T})$

$$B = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

12.5

$X \sim Pois(\lambda)$ — support $0, 1, 2, 3, \ldots$

$\perp$

$Y \sim Bern(p)$ — $0, 1$

$Z = X - Y$

$\longrightarrow \underline{-1, 0, 1, 2, 3, \ldots}$

$$\boxed{\begin{aligned} P(Z=-1) &= P(X=0, Y=1) \\ &= P(X=0)P(Y=1) \\ &= \frac{\lambda^0 e^{-\lambda}}{0!} p = pe^{-\lambda} \end{aligned}}$$

$$\begin{aligned} P(Z=0) &= P(X=0, Y=0) \\ &\quad + P(X=1, Y=1) \\ &= P(X=0)P(Y=0) + P(X=1)P(Y=1) \\ &= \frac{\lambda^0 e^{-\lambda}}{0!}(1-p) + \frac{\lambda^1 e^{-\lambda}}{1!} p \end{aligned}$$

$$X-Y=z \quad = (1-p)e^{-\lambda} + p\lambda e^{-\lambda}$$

$$P(Z=z) = P(X=z+1, Y=1) \qquad z \geq 0$$
$$+ P(X=z, Y=0)$$
$$= P(X=z+1)P(Y=1) + P(X=z)P(Y=0)$$
$$= \frac{\lambda^{z+1} e^{-\lambda}}{(z+1)!} p + \frac{\lambda^{z} e^{-\lambda}}{z!}(1-p)$$

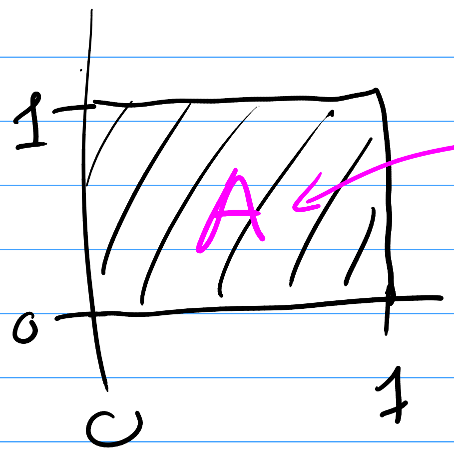
$$Z = \underbrace{X+Y}_{\perp} = X = Z - Y$$

$$P(Z=z) = P(X = \ldots).$$

X, Y unif over unit sq.

0

$$\boxed{f(x,y) = 1 \text{ for } \begin{array}{l} 0<x<1 \\ 0<y<1 \end{array}}$$



$$f(x,y) = \frac{1}{A} = 1$$

$$U = X + Y = g_1(X, Y)$$

$$V = X - Y = g_2(X, Y)$$

1

$$u + v = 2X \Rightarrow \frac{1}{2}(u+v) = X = g_1^{-1}(u,v)$$

$$u - v = 2Y \Rightarrow \frac{1}{2}(u-v) = Y = g_2^{-1}(u,v)$$

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$$J = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\det J = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}$$

$$|\det J| = \frac{1}{2}$$

3

$$f(u,v) = f_{X,Y}\big(\overbrace{g_1^{-1}(u,v)}^{x}, \overbrace{g_2^{-1}(u,v)}^{y}\big)\,|\det J|$$

$$= 1 \cdot \frac{1}{2}$$

for $0 < \frac{1}{2}(u+v) < 1$, $0 < \frac{1}{2}(u-v) < 1$

$$\boxed{f(u,v) = \frac{1}{2} \quad \text{for} \quad \begin{matrix} 0 < u+v < 2 \\ 0 < u-v < 2 \end{matrix}}$$

$-u < v < 2-u$

$u-2 < v < u$