Review: Iterated Expectation

Background: $E[X \mid Y=y] = \int x f(x|y)\,dx = g(y)$

$g: \mathbb{R} \to \mathbb{R}$

Can plug in Y into g, to get $g(Y)$

Notation: $\underbrace{E[X \mid Y] = g(Y)}_{\text{a RV}}$

Iterated Expectation

$$E[X] = E_Y[E[X \mid Y]]$$

Law of Tot. Var

$$Var(X) = E\,Var(X \mid Y) + Var\,E[X \mid Y]$$

Ex. $X \mid Y = y \sim Bin(y, p)$ $p \in [0,1]$

Ex. $X \mid Y = y \sim \text{Bin}(y, p)$, $p \in [0,1]$

$Y \sim \text{Pois}(\lambda)$, $\lambda > 0$

$E[X]$?

① $E[X \mid Y = y] = yp = g(y)$

② $E[X \mid Y] = Yp = g(Y)$

③ $E_Y[E[X \mid Y]]$

$= E[Yp]$

$= pE[Y] = \boxed{p\lambda = E[X].}$

$Var(X)$? $Var(X \mid Y = y) = yp(1-p)$

$Var(X \mid Y) = Yp(1-p)$

$Var(X) = E[Var(Y \mid X)] + Var\, E[Y \mid X]$

$$= E[Y/p(1-p)] + Var(Y/p)$$

$$= p(1-p)E[Y] + p^2 Var(Y)$$

$$= p(1-p)\lambda + p^2\lambda$$

$$= \cdots = \boxed{p\lambda}$$

Some facts:

1. If $\hat{\theta}$ is unbiased for $\tau(\theta)$

$$E[\hat{\theta}] = \tau(\theta).$$

Let W be some function of data (X_ns)

(could be a stat, or not)

Consider

$$\varphi = \varphi(W) = E[\hat{\theta} \mid W]$$

a RV, a fn of W

notice that

$$E[\varphi] = E[E[\hat{\theta}|W]] = E[\hat{\theta}] = \tau(\theta)$$

If φ is a stat, then it is unbiased for $\tau(\theta)$.

(2) $\underline{Var(\varphi) \leq Var(\hat{\theta})}$.

pf: $Var(\hat{\theta}) = Var\, E[\hat{\theta}|W] + E\, Var(\hat{\theta}|W)$

$= Var(\varphi) + (\text{Something} \geq 0)$

So $Var(\hat{\theta}) \geq Var(\varphi)$.

Ex. $X_n \overset{iid}{\sim} N(\theta, 1)$

Let $\hat{\theta} = \frac{1}{2}(X_1 + X_2)$.

$\underline{\text{Note:}}$ $E[\hat{\theta}] = E\left[\frac{1}{2}(X_1 + X_2)\right]$

$$= \frac{1}{2}(EX_1 + EX_2) = \frac{1}{2}2\theta = \theta$$

$$= \tau(\theta).$$

$$Var(\hat{\theta}) = Var\left(\frac{1}{2}(X_1 + X_2)\right)$$

$$= \left(\frac{1}{4}\right)(Var(X_1) + Var(X_2))$$

$$= \left(\frac{1}{4}\right)(1 + 1)$$

$$= \frac{1}{2}.$$

Let $W = X_1$

$$\varphi = E[\hat{\theta} \mid W] = E\left[\frac{1}{2}(X_1 + X_2) \middle| X_1\right]$$

$$= \frac{1}{2}\left(E[X_1 \mid X_1] + E[X_2 \mid X_1]\right)$$

$$= \frac{1}{2}(X_1 + E[X_2])$$

$$= \frac{1}{2}(X_1 + E[X_2])$$

$$\boxed{\varphi = \frac{1}{2}(X_1 + \theta)}$$

Facts:

① $E[\varphi] = \frac{1}{2}\left(E[X_1] + \theta\right) = \frac{1}{2}(\theta + \theta) = \theta$
$= \tau(\theta).$

② $Var(\varphi) = Var\left(\frac{1}{2}(X_1 + \theta)\right)$
$= \frac{1}{4} Var(X_1)$
$= \frac{1}{4}(1)$
$< \frac{1}{2} = Var(\hat{\theta})$

Only problem, φ isn't a stat.

Try again, but use $W = \overline{X}$.

$$\varphi = E[\hat{\theta} \mid \bar{X}]$$

$$= E\left[\tfrac{1}{2}(\cancel{X}_1 + \cancel{X}_2) \mid \bar{X}\right]$$

$$= \tfrac{1}{2} E[\cancel{X}_1 \mid \bar{X}] + \tfrac{1}{2} E[\cancel{X}_2 \mid \bar{X}]$$

$$= \tfrac{1}{2} \cdot 2\, E[X_n \mid \bar{X}]$$

$$= E[\cancel{X}_n \mid \bar{X}]$$

$$= \tfrac{1}{N} N\, E[\cancel{X}_n \mid \bar{X}]$$

$$= \frac{1}{N} \sum_{n=1}^{N} E[\cancel{X}_n \mid \bar{X}]$$

$$= E\left[\frac{1}{N} \sum_n \cancel{X}_n \,\Big|\, \bar{X}\right]$$

$$= E[\bar{X} \mid \bar{X}]$$

$$= \boxed{\bar{X} = \varphi}$$

By our <u>facts</u> :

By our facts:

$$E\varphi = \theta \quad \text{and} \quad Var(\varphi) = \frac{1}{N} < \frac{1}{2} = Var(\hat{\theta}).$$

In this case, $\varphi = \bar{X}$ is a stat!

So φ is a better unbiased est. of θ.

Theorem: Rao-Blackwell Theorem

If $\hat{\theta}$ is unbiased for $\tau(\theta)$ and W is a sufficient stat for θ then

$$\varphi = E[\hat{\theta} \mid W]$$

1. $E\varphi = \tau(\theta)$
2. $Var\,\varphi \leq Var\,\hat{\theta}$
3. φ is a stat. (no θ in its formula)

Pf. of 3

$$\varphi = E[\hat{\theta} | W] = E[\hat{\theta}(\underset{\sim}{X}) | W]$$

$$= \int \underbrace{\hat{\theta}(\underset{\sim}{x})}_{\text{no } \theta} \underbrace{f_{\underset{\sim}{X}|W}(\underset{\sim}{x})}_{\text{no } \theta \text{ b/c } W \text{ is sufficient}} d\underset{\sim}{x}$$

free of θ

Theorem: Lehmann-Scheffe Theorem

tech. conditional

If W is a (complete) sufficient stat for θ and $\hat{\theta}$ is unbiased for $\tau(\theta)$ and $\hat{\theta}$ depends on $\underset{\sim}{X}$ only through W

$$\hat{\theta} = \hat{\theta}(W)$$

then $\hat{\theta}$ is the UMVUE for $\tau(\theta)$.

then θ is the UMVUE for $\tau(\theta)$.

If I can find a fn of my suff. stat. that is unbiased for $\tau(\theta)$ it is the UMVUE.

Ex. Let $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ σ^2 known

Want UMVUE for μ.

Use Lehmann-Scheffe

① Find a SS for μ : $\bar{X}$

② Find unbiased fn of $\bar{X}$:

$E\bar{X} = \mu$ so let $\hat{\mu} = \bar{X}$.

Then $\hat{\mu}$ is the UMVUE.

Ex. Let $\tau(\mu) = \mu^2$.

① Find a SS for μ : $\bar{X}$

(1) Find SS for μ: $\bar{X}$

(2) Find unbiased fn of $\bar{X}$:

Try $\bar{X}^2$:

$$E[\bar{X}^2] = Var(\bar{X}) + E[\bar{X}]^2$$
$$= \left(\sigma^2/N + \mu^2\right) \neq \mu^2$$

Instead try $\bar{X}^2 - \sigma^2/N$

$$E[\bar{X}^2 - \sigma^2/N] = \left(\sigma^2/N + \mu^2\right) - \sigma^2/N$$
$$= \mu^2$$

So let $\hat{\theta} = \bar{X}^2 - \sigma^2/N$ then

(1) $E[\hat{\theta}] = \mu^2$ (unbiased for μ^2)

② it is a fn of data only via SS $\bar{X}$.

So by Lehmann-Scheffe it is the UMVUE.