Lecture 11

Tuesday, October 8, 2024 11:02 AM

Ex. $X_n \overset{iid}{\sim} \text{Bern}(p)$, $p \in [0,1]$

Want UMVUE for $p^2 = \tau(p)$

$\bar{X}^2$? Know: $\bar{X}$ sufficient for p.

$$E[\bar{X}^2] = \text{Var}(\bar{X}) + E[\bar{X}]^2$$

$$= \frac{p(1-p)}{N} + p^2 \neq p^2 \quad (\text{biased})$$

$$= \dots$$

$$= p/N + \frac{N-1}{N} p^2$$

$$E\left[\bar{X}^2 - \frac{\bar{X}}{N}\right] = E[\bar{X}^2] - \frac{E[\bar{X}]}{N}$$

$$= \cancel{\frac{p}{N}} + \frac{N-1}{N} p^2 - \cancel{\frac{p}{N}}$$

$$\boxed{E\left[\frac{N}{N-1}\left(\bar{X}^2 - \frac{\bar{X}}{N}\right)\right] = p^2}$$

So $\frac{N}{N-1}\left(\bar{X}^2 - \frac{\bar{X}}{N}\right)$ is

(1) unbiased for p^2

(2) It is a fn of my suff. stat $\bar{X}$.

So by Lehmann-Scheffe theorem it is the UMVUE for p^2.

Ex. $X_n \overset{iid}{\sim} U(0, \theta)$

what is the UMVUE for $\tau(\theta) = \theta$.

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(1) Find suff. stat. for θ : $X_{(N)}$

(2) Find a fn of $X_{(N)}$ that is unbiased for θ.

Claim: $E[X_{(N)}] = \frac{N}{N+1}\theta$

then $\hat{\theta} = \frac{N+1}{N} X_{(N)}$

is unbiased since

$$E[\hat{\theta}] = \frac{N+1}{N} E[X_{(N)}] = \frac{N+1}{N}\frac{N}{N+1}\theta = \theta$$

and so $\boxed{\hat{\theta} = \frac{N+1}{N} X_{(N)}}$ is the UMVUE for θ.

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pf. of Lehman-Scheffe.

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Will show: if $\hat{\theta}$ is unbiased, a fn of a $\boxed{\text{complete}}$ suff. stat W. Then for any other unbiased est. V

$$\mathrm{Var}(\hat{\theta}) \le \mathrm{Var}(V).$$

Rao-Blackwell thm says that if

$$\varphi = E[V \mid W]$$

then ① $E\varphi = \tau(\theta)$

② $\mathrm{Var}\,\varphi \le \mathrm{Var}(V)$

③ φ is a stat.

then $\mathrm{Var}\,\hat{\theta} \le \mathrm{Var}(V)$

We'll show that $\hat{\theta} = \varphi$.

Consider $\boxed{g(W) = \hat{\theta}(W) - \varphi(W)}$

will show that $\boxed{g \equiv 0 \quad \forall \theta}$

so that $\hat{\theta}(W) - \varphi(W) = 0 \quad \forall \theta$

thus $\hat{\theta}(W) = \varphi(W) \quad \forall \theta$.

Completeness of W:

Say W is <u>complete</u> if

$$E[h(W)] = 0 \iff h \equiv 0$$

h is zero fn

Know:

$$E[g(W)] = E[\hat{\theta}(W) - \varphi(W)]$$

$$= E[\hat{\theta}] - E[\varphi]$$

$$= \tau(\theta) - \tau(\theta)$$

$$= 0 \qquad \forall \theta$$

If W is complete then $g \equiv 0$.

Theorem: UMVUEs are unique

Pf. Let W_1 and W_2 be UMVUEs.

Will show that $W_1 = W_2$.

Consider $W_3 = \frac{1}{2}(W_1 + W_2)$.

Then:

(1) $E[W_3] = \frac{1}{2} E W_1 + \frac{1}{2} E W_2$

$$= \frac{1}{2}\tau(\theta) + \frac{1}{2}\tau(\theta)$$

$$= \tau(\theta)$$

So W_3 is also unbiased for $\tau(\theta)$

$$② \ \mathrm{Var}(W_3) = \mathrm{Var}\left(\frac{1}{2}W_1 + \frac{1}{2}W_2\right)$$

$$= \frac{1}{4}\mathrm{Var}(W_1) + \frac{1}{4}\mathrm{Var}(W_2) + \frac{1}{2}\mathrm{Cov}(W_1, W_2)$$

$$\mathrm{Cor}(W_1, W_2) \leq 1$$

$$\Rightarrow \frac{\mathrm{Cov}(W_1, W_2)}{sd(W_1)\,sd(W_2)} \leq 1$$

$$\Rightarrow \boxed{\mathrm{Cov}(W_1, W_2) \leq sd(W_1)\,sd(W_2)}$$

So

$$\underline{\mathrm{Var}(W_3)} \leq \frac{1}{4}\mathrm{Var}(W_1) + \frac{1}{4}\mathrm{Var}(\overset{W_1}{\cancel{W_2}}) + \frac{1}{2}sd(W_1)\,sd(\overset{W_1}{\cancel{W_2}})$$

$$\leq \frac{1}{4}\mathrm{Var}(W_1) + \frac{1}{4}\mathrm{Var}(W_1) + \frac{1}{2}\mathrm{Var}(W_1)$$

$$\leq \underline{\mathrm{Var}(W_1)}$$

So $Cor(W_1, W_2) = 1$.

This means $W_1 = aW_2 + b$.

However,

$$\underbrace{E(W_1)}_{\tau(\theta)} = a\,\underbrace{E(W_2)}_{\tau(\theta)} + b$$

Must be that $a = 1$, $b = 0$

hence $W_1 = W_2$.

Inequalities

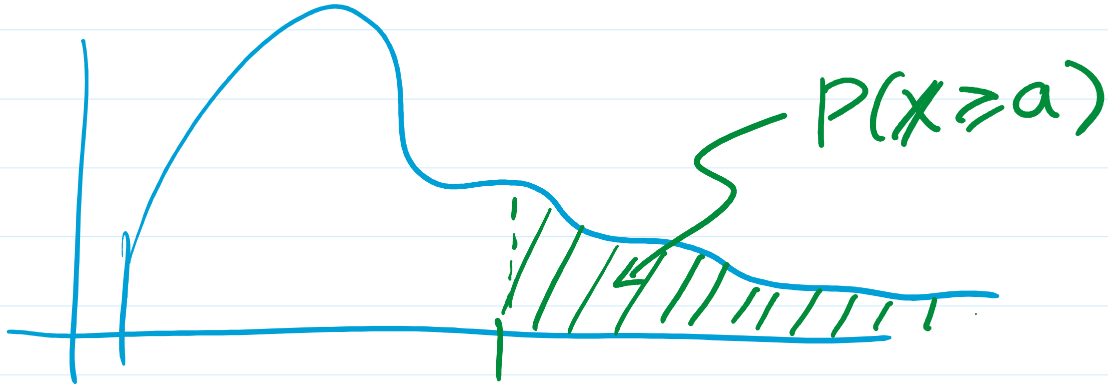
Theorem: Markov's Ineq.

If $X \geq 0$ (support $\subset (0, \infty)$)

then for any $a \geq 0$ we have

Then for any $a > 0$ we have

$$P(X \geq a) \leq \frac{E[X]}{a}.$$



Pf (cts case)

$$E[X] = \int_0^\infty x f(x)\,dx$$

$$= \underbrace{\int_0^a \underbrace{x}_{\geq 0} \underbrace{f(x)}_{\geq 0}\,dx}_{\geq 0} + \int_a^\infty x f(x)\,dx$$

$$\geq \int_a^\infty x f(x)\,dx$$

over (a,∞)
$x \geq a$

$$\geq \int_a^\infty a f(x)\,dx$$

$$\geq \int_a^\infty a f(x)\,dx$$

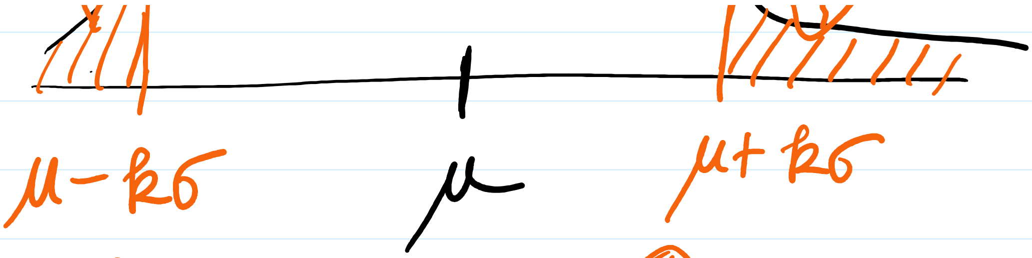
$$= a \int_a^\infty f(x)\,dx$$

$$\boxed{E[X] \geq a\,P(X \geq a)}$$

Theorem: Chebyshev's Ineq.

If X is a RV w/ mean $\mu = E[X]$ and var $\sigma^2 = Var(X)$ then

$$P\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2 .$$



not more than $1/k^2$.

Pf. $Y = \dfrac{(X-\mu)^2}{\sigma^2}$ and $a = k^2$

by Markov's ineq

$$P(Y \geq a) \leq \frac{E(Y)}{a}$$

notice that

$$E[Y] = E\left[\frac{(X-\mu)^2}{\sigma^2}\right]$$

$$= \frac{1}{\sigma^2} \underbrace{E[(X-\mu)^2]}_{Var(X)}$$

$$= \frac{1}{\sigma^2} \sigma^2$$

$$= \frac{1}{\sigma^2} \sigma^2$$
$$= 1$$

thus

$$P(Y \geq a) \leq 1/a$$

and so $\qquad a = k^2 \Rightarrow \sqrt{a} = k$

$$P\left(\frac{(X-\mu)^2}{\sigma^2} \geq a\right) = P\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2.$$

Various Equiv. Vers. of Chebyshev.

① $P\left(\frac{|X-\mu|}{\sigma} \geq k\right) \leq 1/k^2$

② $P\left(\frac{|X-\mu|}{\sigma} < k\right) \geq 1 - 1/k^2$

let $\varepsilon = k\sigma \Leftrightarrow k = \varepsilon/\sigma \Leftrightarrow 1/k^2 = \sigma^2/\varepsilon^2$

③ $P(\ldots) \leq \sigma^2/\varepsilon^2$

③ $P(|X-\mu| \geq \varepsilon) \leq \sigma^2/\varepsilon^2$

④ $P(|X-\mu| < \varepsilon) \geq 1 - \sigma^2/\varepsilon^2$.

<u>Convergence:</u>

Case I : convergence of number $x_n \in \mathbb{R}$

$$\lim_{n\to\infty} x_n = x$$

<u>notation:</u> $x_n \longrightarrow x$

<u>452:</u> $X_n \longrightarrow X$

<u>Recall:</u> $X_n : S \to \mathbb{R}$

for some $s \in S$ we have $X_n(s) \in \mathbb{R}$.

Can define convergence of RVs as convergence of fns.

convergence of fns.

Defn: Pointwise Convergence of Functions

If $(f_n)_{n=1}^{\infty}$ is a seq of fns

$$f_n : \mathbb{R} \to \mathbb{R}$$

and $f : \mathbb{R} \to \mathbb{R}$

we say that the f_n s converge pointwise to f

denoted $f_n \xrightarrow{\text{ptwse}} f$

if $\forall x$

$$f_n(x) \to f(x).$$

Ex. $x = 5$

$$f_1(5), f_2(5), f_3(5), \ldots \to f(5).$$