Defn: Almost Sure Convergence

We say $(X_n)_{n=1}^{\infty}$ converges almost surely if $X_n(s) \to X(s)$ $\forall s \in A \subset S$ where $P(A) = 1$.

Denote this as $X_n \xrightarrow{a.s.} X$.

Ex. $S = [0,1]$ w/ uniform density.

Let $X_n(s) = s + s^n$

$X(s) = s$

Does $X_n \xrightarrow{a.s.} X$?

For which s does $X_n(s) \to X(s)$?

If $s \in [0,1)$ then

as $n \to \infty$

$$X_n(s) = s + s^n \xrightarrow{n} s = X(s).$$

$$X_n(s) = s + s^n \to s = X(s).$$

$$\lim_{n\to\infty} X_n(s) = X(s).$$

If $s = 1$ then

$$X_n(s) = 1 + 1^n = 2 \neq 1 = X(s).$$

So $A = [0,1)$ and $P([0,1)) = 1$

So $X_n \xrightarrow{a.s.} X$.

<u>Defn: Convergence In Probability</u>

We say (X_n) converges in prob. to X

denoted $X_n \xrightarrow{P} X$

if

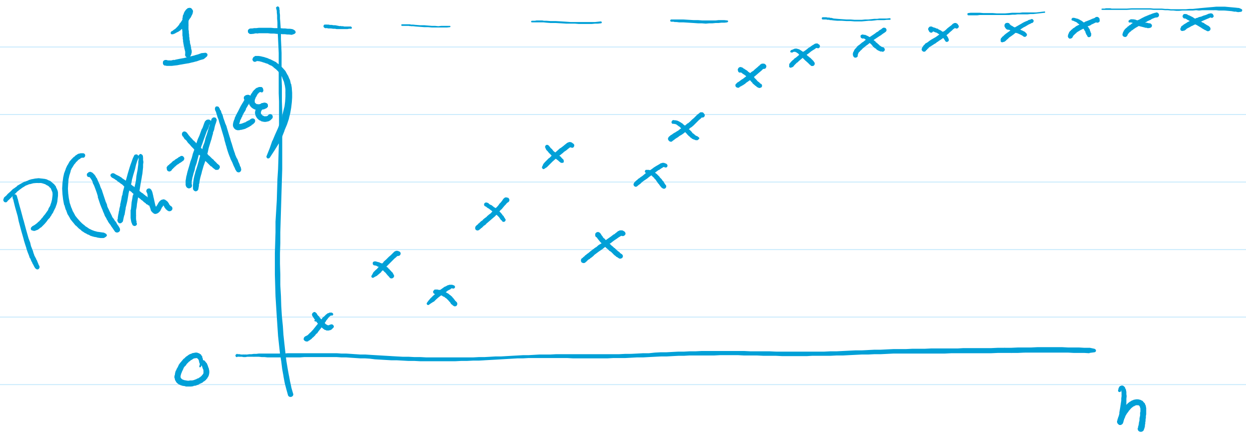
$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}(\underbrace{|X_n - X| < \varepsilon}_{X_n \text{ is less than } \varepsilon \text{ away from } X}) = 1.$$

equiv.

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0.$$

Pick $\varepsilon > 0$, Calc.

$$P(|X_1 - X| < \varepsilon),\ P(|X_2 - X| < \varepsilon),\ \ldots$$



Defn: Convergence In Distribution

We say (X_n) converges in dist to X

(denoted $X_n \xrightarrow{d} X$)

if the CDFs converge (as functions - ptwise)

i.e.

$$F_n \xrightarrow{\text{ptwise}} F$$

CDF of X_n — CDF of X

i.e. $F_n(x) \xrightarrow{n} F(x) \quad \forall x$.

Theorem:

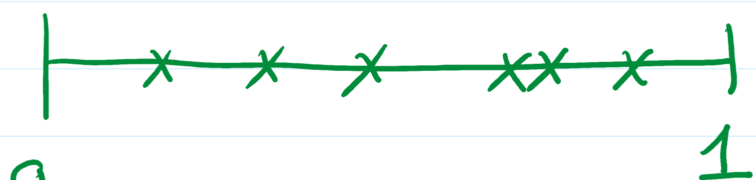
$$a.s. \Rightarrow p \Rightarrow d$$

Converses are typically false.

Ex. Let $X_i \overset{iid}{\sim} U(0,1)$.

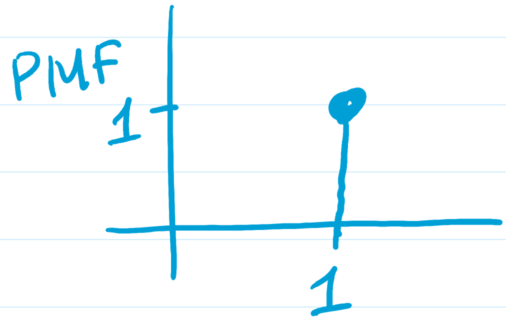
and let $Y_n = \max_{i=1,\dots,n} X_i$.

Intuition:



Y_n creeps up towards 1. as $n \to \infty$

i.e. $Y_n \rightarrow 1$ degenerate RV w/ all mass at 1



Show that $Y_n \xrightarrow{P} 1$.

Need to show: $\forall \varepsilon > 0 \ \ \underbrace{P(|\overset{Y_n}{X_n} - \overset{1}{X}| \geq \varepsilon)}_{\text{as } n\to\infty} \to 0$

$$\boxed{P(|Y_n - 1| \geq \varepsilon)}$$

$$= P(|1 - Y_n| \geq \varepsilon)$$

$$= P(1 - Y_n \geq \varepsilon) \quad \text{b/c } Y_n \leq 1$$

$$= P(Y_n \leq 1 - \varepsilon)$$

$$= P(X_1 \leq 1-\varepsilon, X_2 \leq 1-\varepsilon, \ldots, X_n \leq 1-\varepsilon)$$

$$\text{b/c } Y_n = \max_{i=1,\ldots,n} X_i$$

$$= P(X_1 \leq 1-\varepsilon) \cdot P(X_2 \leq 1-\varepsilon) \cdots P(X_n \leq 1-\varepsilon)$$

$= P(X_1 \le 1-\varepsilon)\cdot P(X_2 \le 1-\varepsilon)\cdots P(X_n \le 1-\varepsilon)$

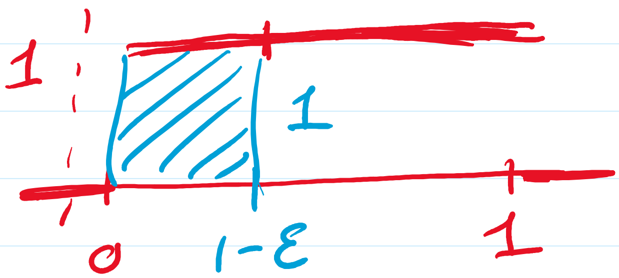
by indep. of X_i

$= P(X_i \le 1-\varepsilon)^n$ b/c X_i have same dist $U(0,1)$

If $\underline{\varepsilon > 1}$ then $1-\varepsilon < 0$

then $P(X_i \le 1-\varepsilon) = 0$

If $\underline{0 < \varepsilon \le 1}$ then



$$P(X_i \le 1-\varepsilon)^n = (1-\varepsilon)^n$$

all together,

$$P(|Y_n - 1| \ge \varepsilon) = \begin{cases} 0, & \varepsilon > 1 \\ (1-\varepsilon)^n, & 0 < \varepsilon \le 1 \end{cases}$$

as $n \to \infty$

$$\begin{cases} 0 & \varepsilon > 1 \end{cases}$$

$$\rightarrow = \begin{cases} 0, & \varepsilon > 1 \\ 0, & 0 < \varepsilon \le 1 \end{cases}$$

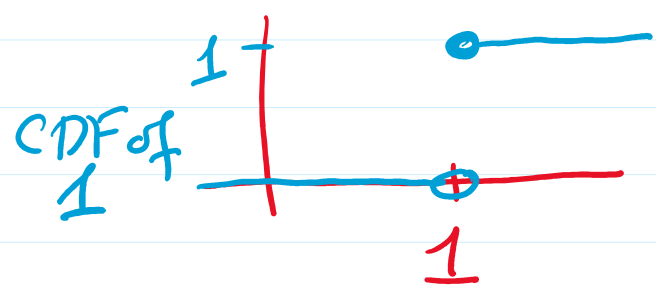
$$= 0$$

So $Y_n \xrightarrow{P} 1$.

Let's show that $Y_n \xrightarrow{d} 1$.

Need to show: $F_n(y) \rightarrow F(y) \quad \forall y$

CDF of Y_n CDF of 1



Let's get CDF of Y_n.

$$F_n(y) = P(Y_n \le y)$$

$$= P\left(\max_{i=1,\dots,n} X_i \le y\right)$$

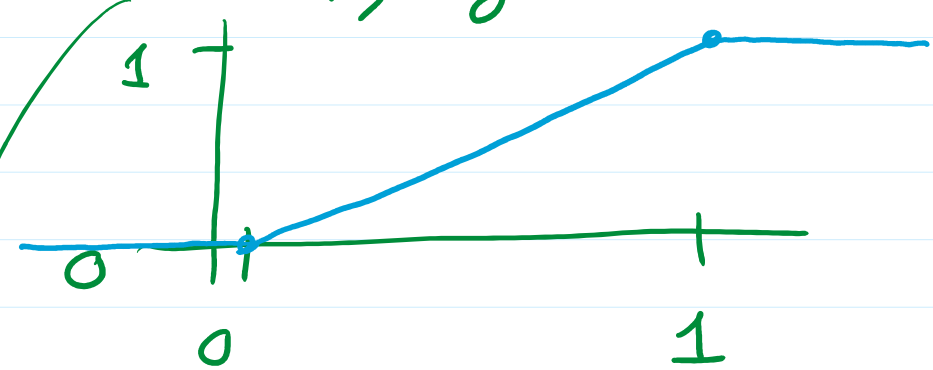
$$= P(X_1 \le y, X_2 \le y, \dots, X_n \le y)$$

$$= P(X_1 \le y)\,P(X_2 \le y)\cdots P(X_n \le y)$$

$$= \boxed{P(X_i \le y)}^n$$

$X_i \overset{iid}{\sim} U(0,1)$ so

$$P(X_i \le y)^n = \begin{cases} 0^n, & y \le 0 \\ y^n, & 0 \le y \le 1 \\ 1^n, & y \ge 1 \end{cases}$$



$$F_n(y) = \begin{cases} 0, & y \le 0 \\ y^n, & 0 \le y \le 1 \\ 1, & y \ge 1 \end{cases}$$

$$F_n(y) = \begin{cases} y, & \ldots \\ 1, & y \geq 1 \end{cases}$$

$$F(y) = \begin{cases} 0, & y < 1 \\ 1, & y \geq 1 \end{cases}$$

$$F_n(y) \xrightarrow{n} \begin{cases} 0, & y < 0 \\ 0, & 0 \leq y < 1 \\ 1, & y \geq 1 \end{cases}$$

$$= F(y)$$

So $Y_n \xrightarrow{d} 1$.

Ex. $X_i \overset{iid}{\sim} U(0,1)$

$$Y_n = \max_{i=1,\ldots,n} X_i$$

$$Z_n = n(1 - Y_n)$$

Let's show that $Z_n \xrightarrow{d} Z$.

Calc.

$$F_n(z) = P(Z_n \le z)$$

$$= P(n(1-Y_n) \le z)$$

$$= P(Y_n \ge 1 - z/n)$$

$$= 1 - P(Y_n < 1 - z/n)$$

$$= 1 - P(X_1 < 1 - z/n) \cdots P(X_n < 1 - z/n)$$

$$= 1 - P(X_i < 1 - z/n)^n$$

$$= \begin{cases} \cancel{1 - 0, \quad 1 - z/n \le 0} \\ \boxed{1 - (1 - z/n)^n, \quad \boxed{0 \le 1 - z/n \le 1}} \\ \cancel{1 - 1, \quad 1 - z/n \ge 1} \end{cases}$$

$$F_n(z) = 1 - \boxed{(1 - z/n)^n} \to e^{-z}$$

as $n \to \infty$

c

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\lim_{n\to\infty} \left(1 + \frac{c}{n}\right)^n = e^c$$

as $n \to \infty$ $\quad \to e^{-z}$ $\qquad \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

$$F_n(z) \to 1 - e^{-z}$$

$$= F(z) \quad \text{for } z > 0$$

$$= \text{CDF of Exp}(1)$$

So, $Z_n \xrightarrow{d} \text{Exp}(1)$.