<u>Ex.</u> Poisson(λ)

Assume $X_n \overset{iid}{\sim} \text{Pois}(\lambda)$ for $\lambda > 0$

$$\boxed{f(x) = \frac{1}{x!}\lambda^{x} e^{-\lambda} \mathbb{1}(x \in \mathbb{N}_0)}$$

$$f(\underset{\sim}{x}) = \prod_{n=1}^{N} f(x_n)$$

$$= \prod_{n=1}^{N} \frac{1}{x_n!} \lambda^{x_n} e^{-\lambda} \mathbb{1}(x_n \in \mathbb{N}_0)$$

$$= \left(\prod_n \frac{1}{x_n!}\right)\left(\prod_n \lambda^{x_n}\right)\left(\prod_n e^{-\lambda}\right)\underbrace{\left(\prod_n \mathbb{1}(x_n \in \mathbb{N}_0)\right)}$$

$$= \underbrace{\left(\prod_n \frac{1}{x_n!}\right)\left(\prod_n \mathbb{1}(x_n \in \mathbb{N}_0)\right)}_{h(\underset{\sim}{x})} \underbrace{\lambda^{\sum_n x_n}} \underbrace{e^{-N\lambda}}_{c(\lambda)}$$

$$\lambda^{\sum_n x_n} = \exp(\log(\lambda^{\sum_n x_n}))$$

$$\lambda^{\sum_n x_n} = \exp\left(\log\left(\lambda^{\sum_n x_n}\right)\right)$$

$$= \exp\Big(\underbrace{\left(\sum_n x_n\right)}_{T(\underset{\sim}{x})}\underbrace{\log(\lambda)}_{\omega(\lambda)}\Big)$$

So this forms an exp. fam.

Short-cut: just need to check marginal

If marginal has the form:

$$f_\theta(x) = h_0(x)c_0(\theta)\exp(T_0(x)\omega_0(\theta))$$

marginal

then

$$f(\underset{\sim}{x}) = \prod_n f(x_n) = \prod_n h_0(x_n)c_0(\theta)\exp(T_0(x_n)\omega_0(\theta))$$

$$= \underbrace{\prod_n h_0(x_n)}_{h(x)}\underbrace{\overbrace{\prod_n c_0(\theta)}^{c_0(\theta)^N}}_{c(\theta)}\exp\Big(\underbrace{\left(\sum_n T_0(x_n)\right)}_{T(x)}\underbrace{\omega_0(\theta)}_{w(\theta)}\Big)$$

$$\underbrace{}_{h(\underset{\sim}{x})} \quad \underbrace{}_{c(\theta)} \qquad \underbrace{}_{T(\underset{\sim}{x})} \quad \underbrace{}_{w(\theta)}$$

this has exp. fam. form.

<u>Punchline:</u> If marginal has this form the joint is an exp fam w/

$$\boxed{\begin{aligned} h(\underset{\sim}{x}) &= \prod_n h_0(x_n) \\ c(\theta) &= c_0(\theta)^N \\ T(\underset{\sim}{x}) &= \sum_n T_0(x_n) \\ w(\theta) &= w_0(\theta) \end{aligned}}$$

<u>Re-do poisson:</u>

$X_n \overset{iid}{\sim} \text{Pois}(\lambda)$

$$f(x_n) = \frac{1}{x_n!} \lambda^{x_n} e^{-\lambda} \mathbb{1}(x_n \in \mathbb{N}_0)$$

$$f(x_n) = \frac{1}{x_n!} \lambda^{x_n} e^{-\lambda} \mathbb{1}(x_n \in \mathbb{N}_0)$$

$$= \underbrace{\left(\frac{1}{x_n!}\right)\left(\mathbb{1}(x_n \in \mathbb{N}_0)\right)}_{h_0(x_n)} \underbrace{e^{-\lambda}}_{c_0(\lambda)} \exp\Big(\underbrace{x_n}_{T_0(x_n)} \underbrace{\log(\lambda)}_{w(\lambda)}\Big)$$

So, yes, this exp. fam, w/ components

$$\begin{cases} h(\underset{\sim}{x}) = \prod_n h_0(x_n) = \prod_n \frac{1}{x_n!} \mathbb{1}(x_n \in \mathbb{N}_0) \\ c(\lambda) = c_0(\lambda)^N = e^{-N\lambda} \\ T(\underset{\sim}{x}) = \sum_n T_0(x_n) = \sum_n x_n \\ w(\lambda) = \log(\lambda) \end{cases}$$

Ex. $X_n \overset{iid}{\sim} U(0, \theta)$.

Q: Is this an exp. fam?

$$f_\theta(x_n) = \frac{1}{\theta} \text{ for } 0 < x_n < \theta$$

$$= \frac{1}{\theta} \underbrace{\mathbb{1}(0 < x_n < \theta)}$$

No way to factor as
(fn of x_n)$\cdot$(fn of θ)

No, not an exp. fam.

General rule: if the unknown param θ shows up in support of dist then its not an exp. fam.

Defn: Sufficiency

Assume $X_n \overset{iid}{\sim} f_\theta$

We say a stat. $T = T(\underset{\sim}{X})$ is sufficient

sufficient

for θ if

$$f_{\underset{\sim}{X} \mid T=t}(\underset{\sim}{x})$$

is "free" of θ.

θ doesn't appear in formula

Ex. If I have data $X_1, \ldots, X_N$ is

$$T = X_1$$

sufficient? No.

Ex. Is $T = \underset{\sim}{X}$ sufficient for θ?

Yes.

Pf. $f(\underset{\sim}{x} \mid T=t)$

$$= \frac{f(\underset{\sim}{x}, t)}{f_T(t)} = \frac{f(\underset{\sim}{x}, \underset{\sim}{x})}{f(\underset{\sim}{x})}$$

no θ

$$= \frac{f(\underset{\sim}{x})}{f(\underset{\sim}{x})} = 1$$

Theorem: Factorization Theorem

T is sufficient for θ

iff

there are fns $g(\theta, T)$ and $h(\underset{\sim}{X})$

so that

$$f_\theta(\underset{\sim}{x}) = g(\theta, \overset{T(\underset{\sim}{X})}{T})\, \underbrace{h(\underset{\sim}{X})}_{\text{fn of } \underset{\sim}{X}}.$$

joint

not θ

not θ

fn of θ and fn of Xs ONLY via T

Ex. Let $X_n \overset{iid}{\sim} \text{Bern}(\theta)$, $\theta \in [0,1]$

Let $T = \frac{1}{N}\sum_{n=1}^{N} X_n = \overline{X}$ = pct of X_ns $= 1$

Apply fact. thrm.

equiv. $f(x_n) = \begin{cases} \theta, & x_n = 1 \\ 1-\theta, & x_n = 0 \end{cases}$

$$f(\underset{\sim}{x}) = \prod_n \theta^{x_n}(1-\theta)^{1-x_n}\,\mathbb{1}(x_n = 0 \text{ or } 1)$$

$$= \left(\prod_n \theta^{x_n}\right)\left(\prod_n (1-\theta)^{1-x_n}\right)\prod_n \mathbb{1}(x_n = 0 \text{ or } 1)$$

$$= \theta^{\sum_n x_n}(1-\theta)^{\sum_n (1-x_n)}\prod_n \mathbb{1}(\cdots)$$

$$= \theta^{N\overline{x}}(1-\theta)^{N-N\overline{x}}\prod_n \mathbb{1}(\cdots)$$

$$= \theta^{N\bar{X}}(1-\theta)^{-N\bar{X}}(1-\theta)^{N}\prod_n \mathbb{1}(\cdots)$$

$$= \underbrace{\left(\frac{\theta}{1-\theta}\right)^{N\overset{T}{\bar{X}}}(1-\theta)^{N}}_{g(\theta,T)=\left(\frac{\theta}{1-\theta}\right)^{NT}(1-\theta)^{N}}\ \underbrace{\prod_n \mathbb{1}(\cdots)}_{h(\underset{\sim}{X})}$$

$$= g(\theta,T)\,h(\underset{\sim}{X}),\quad T=\bar{X}$$

So, T is sufficient.

Ex. Let $X_n \overset{iid}{\sim} U(0,\theta),\ \theta>0$

Find a SS for θ.

1-dim'l

marginal:

$$f_\theta(x_n) = \frac{1}{\theta}\mathbb{1}(0 < x_n < \theta)$$

$$= \frac{1}{\theta}\mathbb{1}(x_n > 0)\mathbb{1}(x_n < \theta)$$

joint:

$$f(\underset{\sim}{x}) = \prod_n f(x_n)$$

$$= \prod_n \frac{1}{\theta}\mathbb{1}(x_n > 0)\mathbb{1}(x_n < \theta)$$

$$= 1/\theta^N \prod_n \mathbb{1}(x_n > 0)\prod_n \mathbb{1}(x_n < \theta)$$

$$= 1/\theta^N \underbrace{\mathbb{1}(x_{(1)} > 0)}_{h(\underset{\sim}{x})}\mathbb{1}(x_{(N)} < \theta)$$

$$= g(\theta, T)\,h(\underset{\sim}{x})$$

$T = X_{(N)}$

$$g(\theta, T) = \frac{1}{\theta^N} \mathbb{1}(T < \theta)$$

So T is sufficient for θ.

Theorem:

If $X_n \overset{iid}{\sim} f_\theta$ and

i.e. is exp. fam.

$$f_\theta(\underset{\sim}{x}) = h(\underset{\sim}{x})\, c(\theta) \exp(T(\underset{\sim}{x})\, w(\theta))$$

then $T(\underset{\sim}{x})$ is sufficient for θ.

Pf: Fact. thrm.

$$f_\theta(\underset{\sim}{x}) = h(\underset{\sim}{x})\, c(\theta) \exp(T(\underset{\sim}{x})\, w(\theta))$$

$\vdots$

?

$$\vdots$$

$$= g(\theta, T)\, h(\underset{\sim}{x})$$

$$g(\theta, T) = c(\theta) \exp(T \cdot w(\theta))$$

Ex. $X_n \overset{iid}{\sim} \text{Pois}(\lambda)$

Prev. showed:

$$f(\underset{\sim}{x}) = \underbrace{\left(\prod_n \frac{1}{x_n!} \mathbb{1}(x_n \in \mathbb{N}_0)\right)}_{h(\underset{\sim}{x})} \overbrace{e^{-N\lambda}}^{c(\lambda)} \exp\left(\overbrace{\left(\sum_n x_n\right)}^{T(\underset{\sim}{x})} \overbrace{\log(\lambda)}^{w(\lambda)}\right)$$

So $T = \sum_n X_n$ is sufficient for λ.

Thrm: any invertible fn of a SS is also sufficient.

E.g. $T = \exp\left(\sum_n X_n\right)$ is sufficient (above)

or $T = \frac{1}{N}\sum_n X_n = \overline{X}$ " "