Ex. $X_n \overset{iid}{\sim} N(\mu, 1)$

Let's find a SS for μ.

(1) Show this an exp. fam.

$$f(x_n) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_n-\mu)^2\right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(x_n{}^2 - 2\mu x_n + \mu^2\right)\right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x_n^2\right) \exp\left(-\frac{1}{2}\mu^2\right) \exp(x_n \mu)$$

$$= h_0(x_n)\, c_0(\mu) \exp\left(T_0(x_n)\, \omega(\mu)\right)$$

So this an exp fam. w/ fns

$$h(\underline{x}) = \prod_n h_0(x_n)$$

$$c(\mu) = c_0(\mu)^N$$

$$\omega(\mu) = \mu$$

$w(\mu) = \mu$

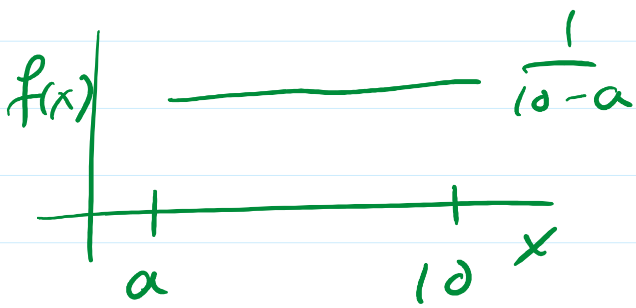
$$\boxed{T(\underset{\sim}{x}) = \sum_n T_0(x_n) = \sum_n x_n}$$

Thus, by prev thm T is sufficient for μ.

Since any 1-1 fn of a SS is sufficient then $\bar{X} = \frac{1}{N} \sum_n x_n$ is sufficient.

__Ex.__ Let $X_n \overset{iid}{\sim} U(a, 10)$ where $0 < a < 10$

Find a SS for a.



Not an exp fam, b/c support deps on unknown a.

Use factorization thrm,

$$f(\underset{\sim}{x}) = \prod_n f(x_n) = \prod_{n=1}^{N} \frac{1}{10-a} \mathbb{I}(a < x_n < 10)$$

$f(\underline{x}) = \prod_n f(x_n) = \prod_{n=1} (10-a$

$$= (10-a)^{-N} \prod_n \mathbb{1}(x_n > a) \prod_n \mathbb{1}(x_n < 10)$$

$$= (10-a)^{-N} \mathbb{1}(x_{(1)} > a) \mathbb{1}(x_{(N)} < 10)$$

$$= g(a, T)\, h(\underset{\sim}{x})$$

$$h(\underline{x}) = \mathbb{1}(x_{(N)} < 10)$$

$$g(a, T) = (10-a)^{-N} \mathbb{1}(T > a)$$

$$T = X_{(1)}$$

So by Fact. thrm, $T = X_{(1)}$ is sufficient.

Defn: Statistic

If $X_n \overset{iid}{\sim} f_\theta$ then a stat. T is a fn of the data (X_ns) whose fmla a fn of the data (X_n's) whose fmla doesn't involve θ. [no θ in fmla]

Ex. $X_n \overset{iid}{\sim} N(\mu, 1)$

$T = \bar{X}$ is a stat. (μ not in fmla)

note: $\bar{X} \sim N(\mu, 1/N)$, not ancillary to μ

however

$T = \bar{X} - \mu$ is not a stat

(μ is in fmla)

$\bar{X} - \mu \sim N(0, 1/N)$, is ancillary to μ

Defn: Ancillary Quantity

$X_n \overset{iid}{\sim} f_\theta$

An ancillary quant. Q is a fn of the data whose dist doesn't involve θ.

[no θ in dist]

Defn: Ancillary Stat.

Defn. Ancillary Stat.

A stat that is ancillary.

[no θ in fmla or dist]

EX. $X_n \overset{iid}{\sim} N(\mu, 1)$

$R = X_{(N)} - X_{(1)}$ ← is a stat, no μ in fmla

note: $X_n \overset{d}{=} \mu + Z_n$ where $Z_n \overset{iid}{\sim} N(0,1)$

by similar logic

$$X_{(N)} \overset{d}{=} \mu + Z_{(N)}$$
$$X_{(1)} \overset{d}{=} \mu + Z_{(1)}$$

So $R = X_{(N)} - X_{(1)} \overset{d}{=} (\cancel{\mu} + Z_{(N)}) - (\cancel{\mu} + Z_{(1)})$

$$\overset{d}{=} Z_{(N)} - Z_{(1)}$$

↑ no μ in dist

So R has no μ in dist.

R is ancillary to μ.

Theorem: Basu's Theorem

If T is a SS for θ and S is an ancillary stat for θ then

$$T \perp S.$$

Theorem from before:

$X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ then $\bar{X} \perp S^2$.

Pf. (1) $\bar{X}$ is sufficient for μ

(2) S^2 is ancillary to μ

why? $\frac{N-1}{\sigma^2} S^2 \sim \chi^2(N-1)$

so $S^2 \sim \frac{\sigma^2}{N-1}\chi^2(N-1)$

no μ!

So, by Basu's thm $\bar{X} \perp S^2$.

Point Estimation

Setup: $X_n \overset{iid}{\sim} f_\theta$ where $\theta \in \Theta$

Defn: A point est. of θ is a stat

$$\hat{\theta} = \hat{\theta}(\underset{\sim}{X})$$

hope: $\hat{\theta} \approx \theta$.

Goals: (1) How do I buid $\hat{\theta}$?

(2) How do I know if $\hat{\theta}$ is good?

First approach: method of moments (MoM)

Defn: the r^{th} moment of a RV X is

Defn: the r^{th} moment of a RV X is

$$\mu_r = E[X^r]$$

Defn: the r^{th} sample moment is

$$m_r = \frac{1}{N}\sum_{n=1}^{N} X_n^r .$$

$$\boxed{\text{idea: } m_r \approx \mu_r .}$$

notice:
$$E[m_r] = \frac{1}{N}\sum_{n=1}^{N} E[X_n^r]$$
$$= \frac{1}{N}\sum_{n=1}^{N} \mu_r$$
$$= \mu_r$$

Ex. $X_n \overset{iid}{\sim} N(\underbrace{\mu, \sigma^2}_{\text{both unknown}})$

Want est $\hat{\mu}$ and $\hat{\sigma}^2$.

Consider (pop) moments

Consider (pop) moments

$$\mu_1 = \mathbb{E}[X_n] = \mu$$

$$\mu_2 = \mathbb{E}[X_n^2] = \text{Var}(X_n) + \mathbb{E}[X_n]^2$$

$$= \sigma^2 + \mu^2$$

Set equal to sample moments:

$$m_1 = \mu_1 \quad \text{and} \quad m_2 = \mu_2$$

and solve for μ and σ^2.

$$\left.\begin{aligned} \overline{X} = \frac{1}{N}\sum_n X_n &= m_1 = \mu_1 = \mu \\ \frac{1}{N}\sum_n X_n^2 &= m_2 = \mu_2 = \mu^2 + \sigma^2 \end{aligned}\right\}$$

Syst of eqns that involve data and μ and σ^2

Solve for μ and σ^2 in terms of X_ns

1. $\boxed{\hat{\mu} = \overline{X}}$ by eqn 1

2. Second eqn says $\frac{1}{N}\sum_n X_n^2 = \mu^2 + \sigma^2$

② second eqn says $\frac{1}{N}\sum_n X_n^2 = \mu^2 + \sigma^2$

plug in $\hat{\mu}$ for μ

$$\frac{1}{N}\sum_n X_n^2 = (\bar{X})^2 + \sigma^2$$

so $$\boxed{\hat{\sigma}^2 = \frac{1}{N}\sum_n X_n^2 - (\bar{X})^2}$$

note: call $\overline{X^2} = \frac{1}{N}\sum_n X_n^2$

then $$\hat{\sigma}^2 = \overline{X^2} - \bar{X}^2$$

$$= \frac{1}{N}\sum_{n=1}^{N}(X_n - \bar{X})^2$$

like sample var but w/ $1/N$.

MoM:

① Find pop. moments $E[X_n^r]$

② set equal to sample moments

$$\frac{1}{N}\sum_n X_n^r$$

③ Solve for params in terms of data.

Ex. Let $X_n \overset{iid}{\sim} Pois(\lambda)$

Get MOM est for λ.

① pop moments: $\mu_1 = E[X_n] = \lambda$

② Set eq to samp. moments

$$\bar{X} = m_1 = \mu_1 = \lambda$$

③ Solve for λ: $\boxed{\hat{\lambda} = \bar{X}}$

Ex. Let $X_n \overset{iid}{\sim} Bin(k, p)$

Lets find MOM est for k and p.

① Get pop moments

$$\mu_1 = E[X_n] = kp$$

$$\mu_2 = E[X_n^2] = Var(X_n) + E[X_n]^2$$

$$= kp(1-p) + k^2p^2$$

② Set eq to sample moments.

$$\overline{X} = m_1 = \mu_1 = kp$$

$$\overline{X^2} = m_2 = \mu_2 = kp(1-p) + k^2p^2$$

③ Solve for k and p in terms of data

$kp = \overline{X}$ so plug into eqn ② to get

$$\overline{X^2} = \overline{X}(1-p) + \overline{X}^2$$

Solve for p,

$$\frac{\overline{X^2} - \overline{X}^2}{\overline{X}} = 1 - p$$

$$\boxed{\hat{p} = 1 - \frac{\overline{X^2} - \overline{X}^2}{\ldots}}$$

can be outside of [0, 1]

$$\boxed{\hat{p} = 1 - \frac{\overline{X^2} - \overline{X}^2}{\overline{X}}}$$

of [0,1]

Eqn (1) said $\overline{X} = kp$

so $\boxed{\hat{k} = \overline{X}/\hat{p}}$

<u>Ex.</u> $X_n \overset{iid}{\sim} U(0,\theta)$

Let's get MOM est for θ.

$$\overline{X} = m_1 = \mu_1 = E[X_n] = \theta/2$$

So solving for θ I get

$$\boxed{\hat{\theta} = 2\overline{X}}$$
