Lecture 8 -- CRLB

Tuesday, September 24, 2024 11:03 AM

<u>Theorem</u>: $I_N(\theta) = N I(\theta)$

Pf. $I_N(\theta) = -E\left[\frac{\partial^2 \ell}{\partial \theta^2}\right]$

$$= -E\left[\frac{\partial^2}{\partial \theta^2} \log f(\underline{X})\right]$$

$$= -E\left[\frac{\partial^2}{\partial \theta^2} \log\left(\prod_n f(X_n)\right)\right]$$

$$= -E\left[\frac{\partial^2}{\partial \theta^2} \sum_n \log f(X_n)\right]$$

$$= \sum_n \underbrace{-E\left[\frac{\partial^2}{\partial \theta^2} \log f(X_n)\right]}_{I(\theta)}$$

$$= N I(\theta).$$

<u>Ex.</u> $X_n \overset{iid}{\sim} \text{Pois}(\lambda)$, $\lambda > 0$

Find $I_N(\lambda)$.

Find $I(\lambda)$ and multiply by N.

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① Find $\log f_\lambda(x)$

$$f_\lambda(x) = \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}(x \in \mathbb{N}_0)$$

$$\log f_\lambda(x) = x \log \lambda - \lambda - \log(x!) + \log \mathbb{1}(x \in \mathbb{N}_0)$$

② Take two derivs w.r.t λ

$$\frac{\partial}{\partial \lambda}[\ldots] = \frac{x}{\lambda} - 1$$

$$\frac{\partial^2}{\partial \lambda^2}[\ldots] = -\frac{x}{\lambda^2}$$

③ Promote x to X and take E

$$I(\lambda) = -E_X\left[\frac{-X}{\lambda^2}\right].$$

Fact: reasonable est. for λ is

$$-\mathbb{E}[X] = \frac{\lambda}{} - \frac{1}{}$$

$$= \frac{1}{\lambda^2} E[X] = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}.$$

est. for λ is $\bar{X}$.

$E\bar{X} = \lambda$ (unbiased)

$Var(\bar{X}) = \frac{\lambda}{N}$

$= 1/I_N(\lambda)$.

④ Multiply by N:

$$\boxed{I_N(\lambda) = N/\lambda}$$

Ex. $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ known

Let's get $I_N(\mu)$.

① $f_\mu(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$

② $\log f_\mu(x) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2$

③ $\frac{\partial}{\partial\mu} \log f_\mu(x) = -\frac{1}{2\sigma^2} 2(x-\mu)(-1)$

$= \frac{1}{\sigma^2}(x-\mu)$

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④ $\frac{\partial^2}{\partial\mu^2}\log f_\mu(x) = -\frac{1}{\sigma^2}.$

⑤ $-E\left[\frac{\partial^2}{\partial\mu^2}\log f_\mu(X)\right] = -E\left[-\frac{1}{\sigma^2}\right] = \frac{1}{\sigma^2} = I(\mu)$

⑥ $I_N(\mu) = N I(\mu) = \frac{N}{\sigma^2}.$

Suspicious, $\bar{X}$ is unbiased for μ.

and $Var(\bar{X}) = \frac{\sigma^2}{N} = \frac{1}{I_N(\mu)}.$

Ex. Revisit poisson but use $\psi = \sqrt{\lambda}$.

$(\lambda = \psi^2)$

Consider getting $I(\psi)$.

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① $f_\psi(x) = \dfrac{\lambda^x e^{-\lambda}}{x!} = \dfrac{(\psi^2)^x e^{-\psi^2}}{x!}$.

② $\log f_\psi(x) = 2x\log(\psi) - \psi^2 - \log(x!)$

③ $\dfrac{\partial}{\partial\psi}[\cdots] = \dfrac{2x}{\psi} - 2\psi$

④ $\dfrac{\partial^2}{\partial\psi^2}[\cdots] = -\dfrac{2x}{\psi^2} - 2$

⑤ $I(\psi) = -E\left[\dfrac{-2X}{\psi^2} - 2\right]$

$= \dfrac{2}{\psi^2}E[X] + 2$

$= \dfrac{2}{\psi^2}\psi^2 + 2 = 4.$

⑥ $I_N(\psi) = 4N.$

Theorem: Fisher Info for Transf.

If $\theta = \tau(\psi)$ $[\Leftrightarrow \psi = \tau^{-1}(\theta)$ if τ invertible$]$

then

$$I(\theta) = \left(\frac{\partial\psi}{\partial\theta}\right)^2 I(\psi)$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

equiv. $I(\psi) = \left(\frac{\partial\theta}{\partial\psi}\right)^2 I(\theta).$

Revisit Ex. $I_N(\lambda) = N/\lambda$

$$\psi = \sqrt{\lambda} \Leftrightarrow \lambda = \psi^2.$$

$$I(\psi) = \left(\frac{\partial\lambda}{\partial\psi}\right)^2 I(\lambda)$$

$$= (2\psi)^2 \, N/\lambda$$

$$= 4\psi^2 \, N/\psi^2$$

$$= 4N.$$

Theorem: Cramér-Rao Lower Bound (CRLB)

If $X_n \overset{iid}{\sim} f_\theta$, $\theta \in \Theta$ and if $\hat{\theta}$ is unbiased for $\tau(\theta)$

⊛ and if f_θ is nice enough

read: 1-dim'l exp. fams.

then
$$\mathrm{Var}(\hat{\theta}) \geq \frac{\left(\frac{\partial \tau}{\partial \theta}\right)^2}{I_N(\theta)} = \underbrace{B}_{\text{Cramér-Rao LB.}}$$

Notes: ① If $\tau(\theta) = \theta$ then $\frac{\partial \tau}{\partial \theta} = 1$

So $B = 1/I_N(\theta)$.

(2) If I can find some θ^* that

(i) unbiased for $\tau(\theta)$

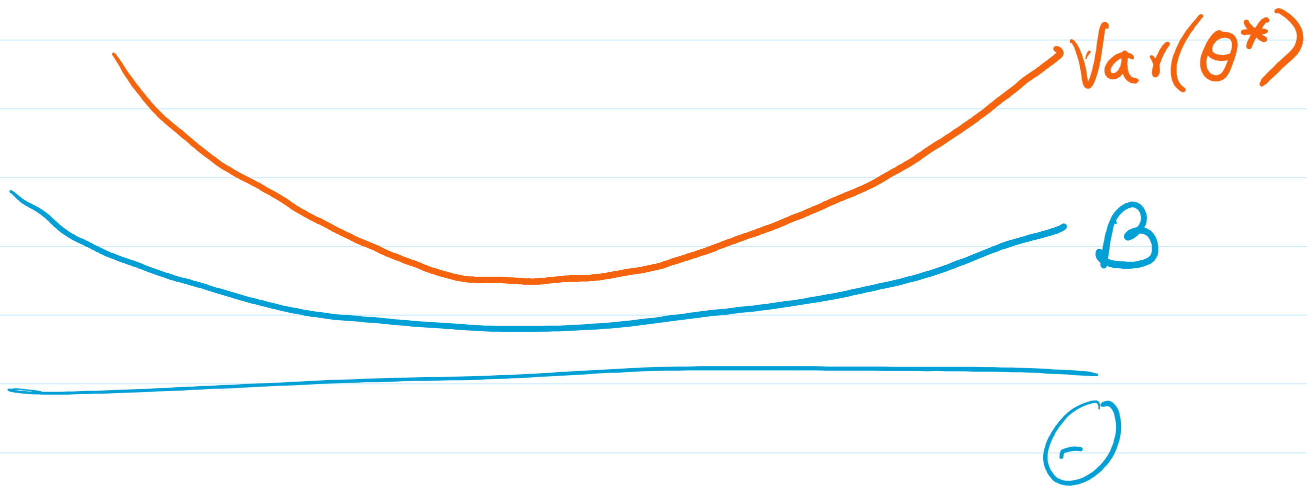
(ii) $Var(\theta^*) = B$

then θ^* is the UMVUE.

(3) If I have some $\hat{\theta}$ that is unbiased for $\tau(\theta)$ but

$$Var(\hat{\theta}) > B.$$

I don't know if $\hat{\theta}$ is the UMVUE.



<u>Ex.</u> $X_n \overset{iid}{\sim} \text{Pois}(\lambda)$ 1-dim'l exp. fam.

Let $\hat{\lambda} = \overline{X}$ and $\tau(\lambda) = \lambda$.

Know $E[\hat{\lambda}] = \lambda$ (unbiased)

and $Var(\hat{\lambda}) = \lambda/N$

and $I_N(\lambda) = N/\lambda$

so $Var(\hat{\lambda}) = 1/I_N(\lambda) = B$

and $\hat{\lambda}$ achieves the CRLB and so it is the UMVUE.

<u>Ex.</u> $X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ known

is a 1-dim'l exp. fam.

" is a exp. fam.

Consider $\hat{\mu} = \bar{X}$ est. $\tau(\mu) = \mu$.

Then ① $E[\hat{\mu}] = \mu$ (unbiased)

② $Var(\hat{\mu}) = \sigma^2/N$

③ $I_N(\mu) = N/\sigma^2$

So $Var(\hat{\mu}) = 1/I_N(\mu) = B$

and thus $\hat{\mu}$ is the UMVUE.

Ex. Consider $\tau(\mu) = \mu^2$.

$$E[\bar{X}^2] = Var(\bar{X}) + E[\bar{X}]^2$$
$$= \sigma^2/N + \mu^2$$

$\hat{\tau} = \bar{X}^2 - \sigma^2/N$ is unbiased for $\tau(\mu)$

$\hat{\tau} = \bar{X}^2 - \sigma^2/N$ is unbiased for $\tau(\mu)$

Claim: $\hat{\tau}$ is the UMVUE but

$$\mathrm{Var}(\hat{\tau}) > \mathrm{CRLB}.$$

Ex. $X_n \overset{iid}{\sim} U(0,\theta)$, $\theta > 0$.

Want the UMVUE for $\tau(\theta) = \theta$.

(1) Unbiased est.

Can show that $E[X_{(N)}] = \frac{N}{N+1}\theta$

so $\hat{\tau} = \frac{N+1}{N} X_{(N)}$ is unbiased for $\tau(\theta) = \theta$.

(2) Calc. Var: $\mathrm{Var}(\hat{\theta}) = \frac{\theta^2}{N(N+2)}$

3) Show $\mathrm{Var}(\hat{\theta}) = B$ (?)

Not 1-dim'l exp. fam.

Need $I(\theta)$

1) $f_\theta(x) = \frac{1}{\theta} \mathbb{1}(0<x<\theta)$

2) $\log f_\theta(x) = -\log(\theta) + \log \mathbb{1}(0<x<\theta)$

3) $\frac{\partial}{\partial\theta}[\cdots] =$

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