Suggested Problems 9

(1) Let $X_n \sim NegativeBinomial(n, \theta)$. Note that $X_n = \sum_{i=1}^n Y_i$ where $Y_i \stackrel{iid}{\sim} Geometric(\theta)$ so that $\mathbb{E}[Y_i] = 1/\theta$ and $Var(Y_i) = \frac{1-\theta}{\theta^2}$. Find some constant c_n so that

$$\sqrt{n}\left(c_n X_n - \frac{1}{\theta}\right) \xrightarrow{d} N\left(0, \frac{1-\theta}{\theta^2}\right).$$

(2) Let $X_n \stackrel{iid}{\sim} Bernoulli(p)$.

(a) For $p \neq 1/2$ show that

$$\sqrt{n}\left(\bar{X}_n(1-\bar{X}_n) - p(1-p)\right) \xrightarrow{d} N\left(0, (1-2p)^2 p(1-p)\right).$$

(b) For $p = 1/2$ show that

$$n\left(\bar{X}_n(1-\bar{X}_n) - p(1-p)\right) \xrightarrow{d} -\frac{1}{4}\chi^2(1).$$

(3) Let $X_i \stackrel{iid}{\sim} Pois(\lambda)$, find a function g so that

$$\sqrt{n}\left(g(\bar{X}_n) - g(\lambda)\right) \xrightarrow{d} N\left(0, 1/4\right).$$

(4) A computer program rounds 12 real numbers to the nearest integer and then adds them. If the error due to each rounding can be assumed to be $Uniform(-.5, .5)$ approximate the probability (using the CLT) that the sum of the rounded numbers differs from the true sum by less than 1. Note that if $Z \sim N(0,1)$ then $P(Z > 1) \approx .16$.

(5) Let $X_n \stackrel{iid}{\sim} Pois(n)$ for $n = 1, 2, 3, \ldots$ What is the limiting distribution of $\frac{X_n - n}{\sqrt{n}}$?